On the Tong-type identity and the mean square of the error term for an extended Selberg class

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Abstract

In 1956, Tong established an asymptotic formula for the mean square of the error term in the summatory function of the Piltz divisor function $d_3(n)$. The aim of this paper is to generalize Tong's method to a class of Dirichlet series that satisfy a functional equation. As an application, we can establish the asymptotic formulas for the mean square of the error terms for a class of functions in the well-known Selberg class. The Tong-type identity and formula established in this paper can be viewed as an analogue of the well-known Voronoï's formula.

Contents

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1 Introduction and main results

Let $a(n)(n \geq 1)$ be a sequence of complex numbers, which is an arithmetic function. One of the most basic goals of analytic number theory is to establish the asymptotic formula for the summatory function of $a(n)$, as accurate as possible. Especially it is important to study the properties of the so-called error term of this asymptotic formula, such as the upper bound, the moments, the sign changes and Ω-results, etc.

One of the important tools in this area is Voronoï's formula of the error term when the sequence $a(n)(n \geq 1)$ satisfy good conditions, for example, they are the coefficients of L-functions of any degree. When $a(n)$ $(n \geq 1)$ are the coefficients of an L-function of degree two, Voronoï's formula of the corresponding error term is a very strong tool to study the properties of the error term. A well-known example is the Dirichlet divisor problem. For a survey of Voronoï's formula, see for example, Steuding [47].

However, when $a(n)$ $(n \geq 1)$ are the coefficients of an L-function of degree ≥ 3, Voronoï's formula is not strong enough to get good results for the properties of the error term. Especially, it fails to give an asymptotic formula for the mean square of the error term even when the degree is 3.

In 1956, Tong [49] established an asymptotic formula for the mean square of the error term in the summatory function of the Piltz divisor function $d_3(n)$. This is the first result in this area for the case of degree 3. Tong's main ingredient is to replace the error term by a corresponding integral such that the difference between the error term and this integral is very small on average.

The aim of this paper is to generalize Tong's method to a class of Dirichlet series that satisfy a functional equation. Especially as an application, we establish the asymptotic formulas for the mean square of the error terms for a class of functions in the well-known Selberg class.

1.1 The Selberg class

To study a summatory function of $a(n)$ and its error term, we consider the Dirichlet series with coefficients $a(n)$ which satisfies a general functional equation. There are many results in this direction, see for example, Chandrasekharan and Narasimhan [3], Redmond [42], Hafner [12, 13], Ivić [20], Meurman [36], Lau [33], Kanemitsu, Sankaranarayanan and Tanigawa [29], Friedlander and Iwaniec [9]. We shall study such a problem for L-functions in the so-called Selberg class.

The well-known Selberg class $\mathcal{S}$ (see for example [26, 44, 46]) consists of non-vanishing Dirichlet series

$$\mathcal{L}(s) := \sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

which satisfies the following hypotheses:

I. **Ramanujan's conjecture**: $a(n) \ll n^{\varepsilon}$ for any $\varepsilon > 0$.

II. **Analytic continuation**: There exists a non-negative integer $m_{\mathcal{L}}$ such that $(s-1)^{m_{\mathcal{L}}}\mathcal{L}(s)$ is an entire function of finite order.

III. **Functional equation**: $\mathcal{L}(s)$ satisfies a functional equation of type

$$\Lambda_{\mathcal{L}}(s) = \omega\overline{\Lambda_{\mathcal{L}}(1-\bar{s})}, \tag{1.1}$$

where

$$\Lambda_{\mathcal{L}}(s) := \mathcal{L}(s)Q^s \prod_{j=1}^{L} \Gamma(\alpha_j s + \beta_j), \tag{1.2}$$

and $Q > 0, |\omega| = 1$ and $\alpha_j > 0, \beta_j \in \mathbb{C}$ with $\operatorname{Re}\beta_j \geq 0$ for all $1 \leq j \leq L$. The number $d = 2\sum_j \alpha_j$ is called the degree of $\mathcal{L}(s)$.

IV. **Euler product**: $\mathcal{L}(s)$ satisfies

$$\mathcal{L}(s) = \prod_p \exp\left(\sum_{n\geq 1} \frac{b(p^n)}{p^{ns}}\right)$$

with suitable coefficients $b(p^n)$ satisfying $b(p^n) \ll p^{nc}$ for some $c < 1/2$.

Many well-known functions are contained in the Selberg class $\mathcal{S}$. We recall some examples. The well-known Riemann zeta-function $\zeta(s)$ and Dirichlet L-functions are functions in $\mathcal{S}$ of degree 1. The product of any ℓ functions in $\mathcal{S}$ of degree 1 is a function in $\mathcal{S}$ of degree ℓ. So $\zeta^2(s)$ is a function in $\mathcal{S}$ of degree 2 and $\zeta^3(s)$ is of degree 3. Let $f(z)$ be

a holomorphic cusp form with respect to $SL_2(\mathbb{Z})$. If $f(z)$ is an eigenform of all Hecke operators, the automorphic L-function attached to $f(z)$ is a function in $\mathcal{S}$ of degree 2. The Dedekind zeta-function over an algebraic number field of degree $\kappa \geq 2$ is a function in $\mathcal{S}$ of degree κ.

The extended Selberg class $\mathcal{S}^{\#}$ (see [26, 27, 28] for an introduction) consists of all Dirichlet series $\sum_{n\geq 1} a(n)n^{-s}$ which satisfy the conditions I*, II and III, where I* means

I*: $\sum_{n\geq 1} a(n)n^{-s}$ is absolutely convergent for $\sigma > 1$.

Consider the sum

$$A(y) := {\sum_{n\leq y}}' a(n).$$

Define $Q(y)$ as the sum of the residues of the function $\mathcal{L}(s)y^s s^{-1}$ and the error term $E(y)$ is defined by

$$E(y) := A(y) - Q(y).$$

Estimates for $E(x)$ and $\int_0^T |E(y)|^2 dy$ were first studied by Voronoï in 1904 and Cramér in 1922 for the special case of the Dirichlet divisor problem respectively, see [52] and [6]. Since then this has been generalized for larger classes of Dirichlet series. There are many results in this direction, see for example, [4, 5, 12, 13, 33, 42, 43].

Roton [43] proved that if $L \in \mathcal{S}^{\#}$ is a function of degree $d \geq 2$. Then

$$E(y) \ll y^{(d-1)/(d+1)+\varepsilon},$$

which is a generalization of Landau's classical result [32]. She also proved that if

$$\sum_{n\leq y} |a(n)|^2 \ll y^{1+\varepsilon},$$

then

$$\int_0^T E^2(y)dy \ll \begin{cases} T^{2-1/d}, & \text{if } 0 < d < 3, \\ T^{3-4/d+\varepsilon}, & \text{if } d \geq 3. \end{cases}$$

1.2 Statements of main results

Suppose $0 \leq \theta < 1$ is a real number. Let $\mathcal{S}^\theta$ denote the set of all Dirichlet series $\mathcal{L}(s) = \sum_{n\geq 1} a(n)n^{-s}$ which satisfy II, III and

$$\text{I}': \quad |a(n)| \ll n^{\theta+\varepsilon}, \quad \sum_{n\leq y} |a(n)|^2 \ll y^{1+\varepsilon}. \tag{1.3}$$

In this paper we assume that $\sum_{j=1}^{L} \beta_j$ is real for the sake of simplicity. Obviously $\mathcal{S} \subset \mathcal{S}^\theta \subset \mathcal{S}^{\#}$. Let $\mathcal{S}^\theta_{real}$ denote the set of all functions in $\mathcal{S}^\theta$ with real coefficients.

Now suppose $a(n)(n \geq 1)$ is a sequence of real numbers such that its corresponding Dirichlet series $\mathcal{L}(s) \in \mathcal{S}^{\theta}_{real}$ and is of degree $d \geq 2$. Without loss of generality, we suppose that d is an integer. Let $1/2 \leq \sigma^* < 1$ denote a fixed real number such that the estimate

$$\int_0^T |\mathcal{L}(\sigma^* + it)|^2 dt \ll T^{1+\varepsilon} \tag{1.4}$$

holds for any $\varepsilon > 0$. We suppose that σ^* satisfies the condition

$$\sigma^* < (d+1)/2d, \tag{1.5}$$

which plays an important role in our paper.

Theorem 1. *Suppose that $d \geq 2$ is a fixed integer, $0 \leq \theta \leq 1/2 - 1/2d$ is a real number. Suppose that $\mathcal{L}(s) \in \mathcal{S}^{\theta}_{real}$ is a function of degree $d \geq 2$ such that (1.4) and (1.5) hold. Then we have*

$$\int_1^T E^2(y) dy = C_d T^{2-1/d} + O(T^{2-\frac{3-4\sigma^*}{2d(1-\sigma^*)-1}+\varepsilon}), \tag{1.6}$$

where $C_d > 0$ is a positive constant.

Corollary 1. *Suppose $0 \leq \theta \leq 1/4$ is a real number and $\mathcal{L}(s) \in \mathcal{S}^{\theta}_{real}$ is of degree 2, then we have*

$$\int_1^T E^2(y) dy = C_2 T^{3/2} + O_\varepsilon(T^{1+\varepsilon}). \tag{1.7}$$

Remark 1. When $\mathcal{L}(s) \in \mathcal{S}^{0}_{real}$ is of degree 2, the result obtained by Voronoï's formula is sometimes stronger. See for example, Meurman [36] or Lau and Tsang [34] in the case of Dirichlet divisor problem. We note that the error term in the asymptotic formula (1.7) is best possible when disregarding ε.

Corollary 2. *Suppose $0 \leq \theta \leq 1/3$ is a real number. Let $\mathcal{L}(s) \in \mathcal{S}^{\theta}_{real}$ is a function of degree 3 such that $\mathcal{L}(s) = \mathcal{L}_1(s)\mathcal{L}_2(s)$, where $\mathcal{L}_1(s) \in \mathcal{S}^{\theta}_{real}$ is a function of degree 1, and $\mathcal{L}_2(s) \in \mathcal{S}^{\theta}_{real}$ is a function of degree 2. Then we have*

$$\int_1^T E^2(y) dy = C_3 T^{5/3} + O_\varepsilon(T^{8/5+\varepsilon}). \tag{1.8}$$

Furthermore if we assume that

$$\int_0^T |\mathcal{L}_2(1/2 + it)|^6 dt \ll T^{2+\varepsilon}, \tag{1.9}$$

then we have

$$\int_1^T E^2(y) dy = C_3 T^{5/3} + O_\varepsilon(T^{14/9+\varepsilon}). \tag{1.10}$$

Remark 2. The assumption (1.9) for degree 2 L-functions is very natural. The first result in this direction is the twelfth power moment of the Riemann zeta-function $\zeta(s)$ (the sixth moment of $\zeta^2(s)$ over the critical line) due to Heath-Brown [16]. Meurmann [38] proved the twelfth power moment for any Dirichlet L-functions. From Heath-Brown [16] and Meurmann [38] we see that (1.9) holds for Dedekind zeta functions of quadratic fields. From [25] we know that (1.9) holds for the automorphic L-functions attached to holomorphic cusp forms.

Remark 3. Tong [49] first established Theorem 1 for the Piltz divisor problem of dimension $d \geq 3$, which becomes a true asymptotic formula when $d = 3$. Fomenko [10] followed Tong's method to study the case of the Dedekind zeta-function of cubic fields.

Theorem 2. *Suppose that $d \geq 2$ is an integer, $0 \leq \theta \leq 1/2 - 1/2d$ is a real number. Suppose that $\mathcal{L}(s) \in \mathcal{S}^{\theta}_{real}$ is a function of degree d such that (1.4) and (1.5) hold. Then the error term $E(t)$ has a distribution function $f(\alpha)$ in the sense that, for any interval $I \subset \mathbb{R}$ we have*

$$T^{-1} mes\{t \in [1, T] : t^{-(d-1)/2d} E(t) \in I\} \to \int_I f(\alpha) d\alpha$$

as $T \to \infty$. The function $f(\alpha)$ and its derivatives satisfy

$$\frac{d^k}{d\alpha^k} f(\alpha) \ll_{A,k} (1 + |\alpha|)^{-A}$$

for $k = 0, 1, 2, \cdots$ and $f(\alpha)$ can be extended to an entire function.

Corollary 3. *Suppose $0 \leq \theta \leq 1/2 - 1/2d$ is a real number. Suppose that $\mathcal{L}(s) \in \mathcal{S}^{\theta}_{real}$ is a function of degree 2, or is a function of degree 3 such that it can be written as a product of a function of degree 1 and a function of degree 2, then Theorem 2 holds.*

Theorem 3. *Suppose that $d \geq 2$ is an integer, $0 \leq \theta \leq 1/2 - 1/2d$ is a real number. Suppose that $\mathcal{L}(s) \in \mathcal{S}^{\theta}_{real}$ is a function of degree d such that (1.4) and (1.5) hold. Then for any real number $0 \leq u \leq 2$, the mean value*

$$\lim_{T \to \infty} T^{-1-\frac{(d-1)u}{2d}} \int_1^T |E(t)|^u dt$$

converges to a finite limit as T tends to infinity.

Corollary 4. *Suppose $0 \leq \theta \leq 1/2 - 1/2d$ is a real number. Suppose that $\mathcal{L}(s) \in \mathcal{S}^{\theta}_{real}$ is a function of degree 2, or is a function of degree 3 such that it can be written as a product of a function of degree 1 and a function of degree 2, Then Theorem 3 holds.*

1.3 A remark on Tong's method

The most important tool up to now to study the behaviour of $E(y)$ is Voronoï's formula, which is usually of the form

$$E(y) = c_1 y^{\frac{d-1}{2d}} \sum_{n \le N} \frac{a(n)}{n^{\frac{d+1}{2d}}} \cos(c_2(ny)^{1/d} + c_3) + O(y^{\frac{d-1}{d}+\varepsilon} N^{-\frac{1}{d}}), \tag{1.11}$$

where c_1, c_2, c_3 are constants and $1 \ll N \ll y$. For a proof of the above general formula, see for example Friedlander and Iwaniec [9]. Roughly speaking, this formula gives a good approximation to the error term, which is usually a finite exponential sum, plus a "permissible small" error.

When $d = 2$, the formula (1.11) is strong enough for us to study the properties of $E(x)$. For example, its upper bound, power moments, sign changes, etc. A good example is the Dirichlet divisor problem, see for example, the survey paper Tsang [50].

However when $d \ge 3$, the error term in (1.11) is always $\gg y^{1-2/d+\varepsilon}$, which is much larger than the expected order $y^{(d-1)/2d}$. So (1.11) can't be used to study the asymptotic behaviour of power moments of $E(y)$, even for the mean square.

In [48, 49], Tong developed a method to study the mean square of the error term in the summatory function of the Piltz divisor function $d_\ell(n)$, which denotes the number of ways such that n can be written as a product of ℓ natural numbers. When $\ell = 3$, Tong's method gives a true asymptotic formula of the mean square of the error term $\Delta_3(y)$. Tong's main ingredient is to replace the error term by a corresponding integral such that the difference between the error term and this integral is very small on average.

The aim of this paper is to generalize Tong's method to the general case. Actually we shall generalize Tong's method to a class of functions much more general than $\mathcal{S}^\theta_{real}$. As applications, we establish the asymptotic formula of the mean square of $E(y)$ for functions in $\mathcal{S}^\theta_{real}$.

1.4 Organization of this paper

The organization of this paper is as follows. In Section 2 we shall introduce a class of more general functions. In Section 3 we give some preliminary lemmas about the so-called generalized Bessel functions. In Section 4 we shall establish a Tong-type identity for the corresponding error term $E(y)$. As some simple applications of the Tong's identity, in Section 5 we give a lower bound for integrals involving $E(y)$ and study the small values of $E(y)$. In Section 6 we shall establish a truncated Tong-type formula for an integral involving $E(y)$. In Section 7 we shall estimate some exponential integrals, which are important in the proof. In Section 8 we shall prove Theorem 1. In Section 9 we shall prove Theorem 2 and Theorem 3. In Section 10, we give some examples.

2 A class of more general arithmetic functions

For future applications in mind (e.g. [2]), we shall derive a Tong-type identity of the error term for a more general class of functions than that of Selberg class. In this section, following basically to Chandrasekharan and Narasimhan [3, 4] and Hafner [12], we introduce such a class of functions.

Definition. Let $\{a(n)\}$ and $\{b(n)\}$ be two sequences of complex numbers, not identically zero. Let $\{\lambda_n\}$ and $\{\mu_n\}$ be two strictly increasing sequences of positive numbers tending to infinity. Suppose that the series

$$\varphi(s) = \sum_{n=1}^{\infty} a(n)\lambda_n^{-s}, \quad \psi(s) = \sum_{n=1}^{\infty} b(n)\mu_n^{-s}$$

converge in some half-plane and have abscissae of absolute convergence σ_a^* and σ_b^*, respectively. Define two gamma factors

$$\Delta_1(s) = \prod_{j=1}^{N} \Gamma(\alpha_j s + \beta_j) \tag{2.1}$$

and

$$\Delta_2(s) = \prod_{h=1}^{N'} \Gamma(\alpha'_h s + \beta'_h), \tag{2.2}$$

where α_j and α'_h are positive real numbers and β_j and β'_h are complex numbers. We assume throughout that

$$\alpha := \sum_{j=1}^{N} \alpha_j = \sum_{h=1}^{N'} \alpha'_h. \tag{2.3}$$

Let r be real. We say that φ and ψ satisfy the functional equation

$$\Delta_1(s)\varphi(s) = \Delta_2(r-s)\psi(r-s) \tag{2.4}$$

if there exists in the s-plane a domain D that is the exterior of a compact set $\mathfrak{S}$ (we call it a "singularity set") and on which there exists a holomorphic function $\chi(s)$ ($s = \sigma + it$, σ and t real) such that

$$\text{(i)} \quad \lim_{|t|\to\infty} \chi(\sigma + it) = 0$$

uniformly in every interval $-\infty < \sigma_1 \le \sigma \le \sigma_2 < +\infty$, and

$$\text{(ii)} \quad \chi(s) = \begin{cases} \Delta_1(s)\varphi(s) & \text{for } \sigma > \sigma_a^*, \\ \Delta_2(r-s)\psi(r-s) & \text{for } \sigma < r - \sigma_b^*. \end{cases}.$$

Finally suppose that both $\varphi(s)$ and $\psi(s)$ have only a finite number of poles on the complex plane $\mathbb{C}$.

Remark 4. When $\Delta_1(s) = \Delta_2(s)$, we get the class of functions defined in Chandrasekharan and Narasimhan [3, 4]. Clearly any function in the extended Selberg class $\mathcal{S}^{\#}$ satisfies the functional equation of the form (2.4).

Remark 5. There are many other arithmetic functions in the above class but $\Delta_1(s) \neq \Delta_2(s)$. One example is the well-known asymmetric many-dimensional divisor problem. For a survey of the asymmetric many-dimensional divisor problem, see Krätzel [31] or Ivić etc. [23]. A forthcoming paper [2] in this direction is in preparation through the approach of this paper.

From now on we always assume that $a(n), b(n)$ are real. We suppose that there are infinitely many n such that $|a(n)| \gg 1$ and the same holds for $b(n)$.

For $y > 0$ and a real number ϱ, we define the summatory function $A_\varrho(y)$ of the arithmetical function $a(n)$ by

$$A_\varrho(y) = \frac{1}{\Gamma(\varrho+1)} {\sum_{\lambda_n \le y}}' a(n)(y-\lambda_n)^\varrho,$$

where the symbol $'$ indicates that the last term has to be halved if $\varrho = 0$ and $y = \lambda_n$. When ϱ is negative, $A_\varrho(x)$ is defined only for those positive y not equal to any λ_n.

Let $s_0 = \sup\{|s| : s \in \mathfrak{S}\}$, where $\mathfrak{S}$ is the "singularity set" in Definition, and $t_0 = \max\{|\frac{\beta_j}{\alpha_j}|, |\frac{\beta_h'}{\alpha_h'}| : j = 1, 2, \dots, N, h = 1, 2, \dots, N'\}$. Choose two constants $c > \max\{\sigma_a^*, \sigma_b^*, s_0, t_0\}$ and $R > \max\{s_0, t_0\}$. Choose the third constant $b \notin \mathbb{Q}$ such that $r - b$ is not an integer and $b > \max\{c, r\}$. We choose $a \le \min\{\sigma_b^* - \frac{2}{\alpha}, \frac{r}{2} - \frac{1}{2\alpha}\}$. In fact a will be chosen to be small so that the relevant integral converges absolutely.

Let $\mathcal{C}$ be the rectangle with vertices $c \pm iR$ and $r - b \pm iR$, taken in the counter-clockwise direction. For convenience, we shall use $\mathcal{C}_{u,v}$ to denote the oriented polygonal path with vertices $u - i\infty, u - iR$, $v - iR, v + iR, u + iR$, and $u + i\infty$ in this order.

We define the residual function or the main term

$$Q_\varrho(y) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\Gamma(s)\varphi(s) y^{\varrho+s}}{\Gamma(s+\varrho+1)} ds. \tag{2.5}$$

From our choices of b, c and R, the path $\mathcal{C}$ encircles all of S. By the residue theorem we have

$$Q_\varrho(y) = \sum_{s_j} y^{\varrho+s_j} P_{s_j}(\log y), \tag{2.6}$$

where s_j runs over all the poles of $\Gamma(s)\varphi(s)y^{\varrho+s}/\Gamma(s+\varrho+1)$ inside $\mathcal{C}$, $P_{s_j}(t)$ is a polynomial of t such that its degree is the order of the pole s_j minus 1.

We define the error term in the asymptotic formula for $A_\varrho(y)$ as

$$E_\varrho(y) = A_\varrho(y) - Q_\varrho(y).$$

As for $\psi(s)$ we define similarly that

$$A_\varrho^*(y) = \frac{1}{\Gamma(\varrho+1)} {\sum_{\mu_n \le y}}' b(n)(y-\mu_n)^\varrho,$$

$$Q^*_\varrho(y) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\Gamma(s)\psi(s)y^{\varrho+s}}{\Gamma(s+\varrho+1)}ds,$$
$$E^*_\varrho(y) = A^*_\varrho(y) - Q^*_\varrho(y).$$

Similarly to (2.6) we have

$$Q^*_\varrho(y) = \sum_{s_j} y^{\varrho+s_j} P^*_{s_j}(\log y),$$

where s_j runs over all the poles of $\Gamma(s)\psi(s)y^{\varrho+s}/\Gamma(s+\varrho+1)$ inside $\mathcal{C}$, $P^*_{s_j}(t)$ is a polynomial such that its degree is the order of the pole s_j minus 1.

When $\varrho = 0$, for simplicity, we also use the following notation

$$A(y) = A_0(y), \qquad Q(y) = Q_0(y), \qquad E(y) = E_0(y),$$
$$A^*(y) = A^*_0(y), \qquad Q^*(y) = Q^*_0(y), \qquad E^*(y) = E^*_0(y).$$

We suppose that $\sigma^* < \sigma^*_b$ is a real number such that the estimate

$$\int_{-T}^{T} |\psi(\sigma^* + it)|^2 dt \ll T^{1+\varepsilon} \tag{2.7}$$

holds. Since there are infinitely many n such that $b(n) \gg 1$, we have $\sigma^* \geq 0$. We also suppose that $\psi(s)$ doesn't have any poles in the region $\sigma \leq \sigma^*$.

3 Some preliminary lemmas

We shall prove a fundamental lemma of the asymptotic expansion of the integral which we need later. First we recall the well-known Stirling's formula for gamma function.

Lemma 1 (Stirling's formula)**.** *Let c be a constant. Then there exist some constants γ_j and γ'_j depending on c such that for any positive integer m,*

$$\log\Gamma(s+c) = \left(s+c-\frac{1}{2}\right)\log s - s + \frac{1}{2}\log 2\pi + \sum_{j=1}^{m}\frac{\gamma_j}{s^j} + O\left(\frac{1}{|s|^{m+1}}\right) \tag{3.1}$$

and

$$\Gamma(s+c) = \sqrt{2\pi}e^{\left(s+c-\frac{1}{2}\right)\log s - s}\left(1 + \sum_{j=1}^{m}\frac{\gamma'_j}{s^j} + O\left(\frac{1}{|s|^{m+1}}\right)\right) \tag{3.2}$$

uniformly for $|\arg s| < \pi - \delta$ for fixed $\delta > 0$, as $|s| \to \infty$.

The constants γ_j in (3.1) are given explicitly by

$$\gamma_j = \frac{(-1)^{j-1}B_{j+1}(c)}{j(j+1)},$$

where $B_j(x)$ is the Bernoulli polynomials of degree j. See for example, Wang and Guo [54, p. 123 (5)] and see also A. Erdélyi et al. [8].

For two gamma factors $\Delta_1(s)$ and $\Delta_2(s)$ defined by (2.1) and (2.2), respectively, we define

$$\mu = \sum_{j=1}^{N}\left(\beta_j - \frac{1}{2}\right) + \frac{1}{2}, \qquad \mu' = \sum_{h=1}^{N'}\left(\beta'_h - \frac{1}{2}\right) + \frac{1}{2}, \tag{3.3}$$

$$\nu = \sum_{j=1}^{N}\left(\beta_j - \frac{1}{2}\right)\log\alpha_j, \qquad \nu' = \sum_{h=1}^{N'}\left(\beta'_h - \frac{1}{2}\right)\log\alpha'_h, \tag{3.4}$$

$$\tau = \sum_{j=1}^{N}\alpha_j\log\alpha_j, \qquad \tau' = \sum_{h=1}^{N'}\alpha'_h\log\alpha'_h. \tag{3.5}$$

Furthermore we define

$$\theta_\varrho = \frac{r}{2} - \frac{1}{4\alpha} + \varrho\left(1 - \frac{1}{2\alpha}\right) + \frac{\mu' - \mu}{2\alpha} \tag{3.6}$$

and

$$h = 2\alpha e^{-\frac{\tau+\tau'}{2\alpha}}.$$

Let $\mathcal{C}_{a,b}$ be a curve defined in the previous section such that all poles of $\Delta_2(s)$ lie in the left hand side of $\mathcal{C}_{a,b}$. If β_j and β'_h are real, this condition means that b is greater than the maximal pole of $\Delta_2(s)$.

Lemma 2. *We assume that*

$$\overline{\Delta_1(s)} = \Delta_1(\bar{s}) \quad \textit{and} \quad \overline{\Delta_2(s)} = \Delta_2(\bar{s}). \tag{3.7}$$

Let ω be a real number and M an integer. Let a be a real number such that

$$a < \frac{r}{2} - \frac{\mu' - \mu + \omega + M + 1}{2\alpha}. \tag{3.8}$$

Let $\mathcal{D}$ be a domain such that

$$\mathcal{D} = \mathbb{C}\backslash\{s \in \mathbb{C} \mid \operatorname{Re} s < b', |\operatorname{Im} s| < R'\}$$

with some constant b' and R'. We suppose that $\mathcal{D} \supset \mathcal{C}_{a,b}$, and furthermore if $\omega \neq 0$ we assume that $b' > 0$. Suppose that a function $F(s)$ is regular in the domain $\mathcal{D}$ and has an asymptotic expansion

$$F(s) \sim s^M \sum_{n=0}^{\infty}\frac{d_n}{s^n} \quad (d_0 \neq 0) \tag{3.9}$$

with real coefficients d_n *as* $|s| \to \infty$. *Let* $\mathcal{I}(x)$ *be the function defined by*

$$\mathcal{I}(x) := \int_{\mathcal{C}_{a,b}} \frac{\Delta_2(s)}{\Delta_1(r-s)} s^{\omega} F(s) x^{-s} ds. \tag{3.10}$$

Then for any positive integer m *we have an asymptotic expansion*

$$\mathcal{I}(x) = 2iC \sum_{k=0}^{m} c_k x^{\frac{\tilde{M}-k+\frac{1}{2}}{2\alpha}} \cos\left(hx^{\frac{1}{2\alpha}} - a_k \pi\right) + O\left(x^{\frac{\tilde{M}-m-\frac{1}{2}}{2\alpha}}\right),$$

where

$$\begin{aligned} \tilde{M} &= \mu' - \mu - \alpha r + \omega + M, \\ a_k &= \frac{1}{2}\left(M + \omega - k + \alpha r + \mu + \mu' - \frac{1}{2}\right), \\ C &= (2\pi)^{\frac{N'-N}{2}} e^{\nu' - \nu - \tau r - \frac{(\tau+\tau')(\tilde{M}+\frac{1}{2})}{2\alpha}} \end{aligned} \tag{3.11}$$

and c_k *are real constants which do not depend on* x. *In particular*

$$c_0 = \frac{\sqrt{\pi}}{\sqrt{\alpha}} d_0$$

and

$$c_1 = \frac{\sqrt{\pi}}{\sqrt{\alpha}} \left\{ d_0 \left(-\frac{1}{2\alpha}\left(\frac{\tilde{M}^2}{2} - \frac{1}{24}\right) + \frac{1}{2}\left(\sum_{h=1}^{N'} \frac{B_2(\beta'_h)}{\alpha'_h} + \sum_{j=1}^{N} \frac{B_2(\alpha_j r + \beta_j)}{\alpha_j} \right) \right) + d_1 \right\},$$

where $B_2(x)$ *is the Bernoulli polynomial of degree 2.*

Remark 6. This lemma is a generalization of Theorem 3 in Tong [48]. But Tong omitted the proof of his Theorem 3.

Proof. We note first that the integral (3.10) is convergent absolutely under the condition (3.8).

We derive an asymptotic expansion of the integrand of (3.10). Here for simplicity we use the symbol $\sim$ to denote that the right hand side of $\sim$ is an asymptotic expansion of a function in the left hand side.

Consider $\Delta_2(s)$ first. By (3.2), there exist constants b'_h such that

$$\Delta_2(s) = \prod_{h=1}^{N'} \Gamma(\alpha'_h s + \beta'_h) \sim (2\pi)^{N'/2} \exp(g_2(s)) \left(1 + \frac{b'_1}{s} + \frac{b'_2}{s^2} + \cdots\right),$$

where $g_2(s)$ is given by

$$g_2(s) = \sum_{h=1}^{N'} \left\{ \left(\alpha'_h s + \beta'_h - \frac{1}{2}\right)(\log s + \log \alpha'_h) - \alpha'_h s \right\}$$

$$= \left(\alpha s + \mu' - \frac{1}{2}\right) \log s + (\tau' - \alpha)s + \nu'.$$

Similarly we have, with some constants b_j'',

$$\begin{aligned}\Delta_1(r-s) &= \prod_j \Gamma(-\alpha_j s + \alpha_j r + \beta_j) \\ &\sim (2\pi)^{N/2} \exp(g_1(s)) \left(1 + \frac{b_1''}{s} + \frac{b_2''}{s^2} + \cdots\right),\end{aligned}$$

where $g_1(s)$ is given by

$$\begin{aligned}g_1(s) &= \sum_{j=1}^{N} \left\{ \left(-\alpha_j s + \alpha_j r + \beta_j - \frac{1}{2}\right)(\log(-s) + \log \alpha_j) + \alpha_j s \right\} \\ &= \left(-\alpha s + \alpha r + \mu - \frac{1}{2}\right) \log(-s) - (\tau - \alpha)s + \tau r + \nu.\end{aligned}$$

In particular

$$b_1' = \frac{1}{2} \sum_{h=1}^{N'} \frac{B_2(\beta_h')}{\alpha_h'} \qquad \text{and} \qquad b_1'' = -\frac{1}{2} \sum_{j=1}^{N} \frac{B_2(\alpha_j r + \beta_j)}{\alpha_j}.$$

By using

$$\log(-s) = \log s - \pi i \operatorname{sgn}(t)$$

for non-zero and non-negative s, where $t = \operatorname{Im} s$, we can see easily that

$$\begin{aligned}g(s) :&= g_2(s) - g_1(s) \\ &= (2\alpha s + \mu' - \mu - \alpha r) \log s - \pi i \alpha s \operatorname{sgn}(t) \\ &\quad + \pi i \left(\alpha r + \mu - \frac{1}{2}\right) \operatorname{sgn}(t) + (\tau + \tau' - 2\alpha)s + \nu' - \nu - \tau r.\end{aligned}$$

Combing these formulas we find that with some constants p_n,

$$\begin{aligned}&\frac{\Delta_2(s)}{\Delta_1(r-s)} s^{\omega} F(s) x^{-s} \\ &\sim (2\pi)^{(N'-N)/2} \exp(g(s)) e^{-s \log x} \left(1 + \frac{b_1'}{s} + \cdots\right) \left(1 + \frac{b_1''}{s} + \cdots\right)^{-1} \\ &\qquad \times s^{\omega + M} \left(d_0 + \frac{d_1}{s} + \cdots\right) \\ &= (2\pi)^{(N'-N)/2} \exp(g(s)) e^{-s \log x} s^{\omega + M} \sum_{n=0}^{\infty} \frac{p_n}{s^n}.\end{aligned}$$

By (3.7) and the assumption that d_n are real, we have

$$\overline{\int_{\mathcal{C}_{a,b}\cap\{s|t>0\}} \frac{\Delta_2(s)}{\Delta_1(r-s)} s^{\omega} F(s) x^{-s} ds} = -\int_{\mathcal{C}_{a,b}\cap\{s|t<0\}} \frac{\Delta_2(s)}{\Delta_1(r-s)} s^{\omega} F(s) x^{-s} ds,$$

hence in order to evaluate $\mathcal{I}(x)$ it is enough to consider the case $t > 0$ and take the imaginary part. So we suppose that $t > 0$. Let

$$f(s) = 2\alpha s \log s + (\tau + \tau' - 2\alpha)s - \pi i \alpha s - s \log x.$$

Then

$$\frac{\Delta_2(s)}{\Delta_1(r-s)} s^{\omega} F(s) x^{-s} \sim C_0 e^{\pi i(\alpha r + \mu - 1/2)} s^{\mu' - \mu - \alpha r + \omega + M} \exp(f(s)) \sum_{n=0}^{\infty} \frac{p_n}{s^n} \tag{3.12}$$

where we put

$$C_0 = (2\pi)^{(N'-N)/2} e^{\nu' - \nu - \tau r}.$$

Now let

$$y := x e^{-(\tau+\tau')}.$$

We define the new parameters w, ξ and η by

$$\begin{aligned} s &= y^{\frac{1}{2\alpha}} w \\ w &= i(1+\xi), \quad |\xi| \text{ small} \\ \eta^2 &= -2\alpha i \left((1+\xi)\log(1+\xi) - \xi\right). \end{aligned}$$

The branch of η will be taken as

$$\xi = \xi(\eta) = \frac{i^{1/2}\eta}{\sqrt{\alpha}} + \frac{1}{6}\left(\frac{i^{1/2}\eta}{\sqrt{\alpha}}\right)^2 - \frac{1}{72}\left(\frac{i^{1/2}\eta}{\sqrt{\alpha}}\right)^3 + \cdots.$$

Let δ be a small constant. We put

$$w_1 = u_1 + iv_1 = i(1+\xi(-\delta)) \quad w_2 = u_2 + iv_2 = i(1+\xi(\delta)).$$

Define the paths of integration by

$$\begin{aligned} \mathcal{L}_0 &= \left\{ s \mid s = y^{\frac{1}{2\alpha}} i(1+\xi(\eta)), -\delta \le \eta \le \delta \right\} \\ \mathcal{L}_1 &= \left\{ s \mid s = y^{\frac{1}{2\alpha}} (u_1 + iv), 0 \le v \le v_1 \right\} \\ \mathcal{L}_2 &= \left\{ s \mid s = y^{\frac{1}{2\alpha}} (u_2 + iv), v \ge v_2 \right\}. \end{aligned}$$

By Cauchy's theorem and the remark above,

$$\mathcal{I}(x) = 2i \sum_{j=0}^{2} \operatorname{Im} \int_{\mathcal{L}_j} \frac{\Delta_2(s)}{\Delta_1(r-s)} s^{\omega} F(s) x^{-s} ds.$$

First we consider the integral

$$J_0 = \int_{\mathcal{L}_0} \frac{\Delta_2(s)}{\Delta_1(r-s)} s^{\omega} F(s) x^{-s} ds.$$

From the above choice of parameters, we have

$$\begin{aligned} f(s) &= 2\alpha s \log s - 2\alpha s - \pi i \alpha s - s \log y \\ &= 2\alpha y^{\frac{1}{2\alpha}} i(1+\xi)\Big(\log(1+\xi) - 1\Big) \\ &= -2\alpha y^{\frac{1}{2\alpha}} i + 2\alpha y^{\frac{1}{2\alpha}} i\Big((1+\xi)\log(1+\xi) - \xi\Big). \end{aligned}$$

Therefore we have

$$\frac{\Delta_2(s)}{\Delta_1(r-s)} s^{\omega} F(s) x^{-s} = C_0 e^{-2\alpha i y^{\frac{1}{2\alpha}}} e^{\pi i(\alpha r + \mu - 1/2)} \exp(-y^{\frac{1}{2\alpha}} \eta^2) \sum_{n=0}^{\infty} p_n s^{\tilde{M}-n},$$

where $\tilde{M}$ is defined by (3.11). In terms of the parameter η, we have

$$\begin{aligned} &\sum_{n=0}^{\infty} p_n s^{\tilde{M}-n} = \sum_{n=0}^{\infty} p_n \left(i y^{\frac{1}{2\alpha}}(1+\xi)\right)^{\tilde{M}-n} \\ &= \sum_{n=0}^{\infty} p_n (i y^{\frac{1}{2\alpha}})^{\tilde{M}-n} \left(1 + (\tilde{M}-n)\xi + \frac{(\tilde{M}-n)(\tilde{M}-n-1)}{2}\xi^2 + \cdots\right) \\ &= \sum_{n=0}^{\infty} p_n (i y^{\frac{1}{2\alpha}})^{\tilde{M}-n} \left(1 + (\tilde{M}-n)\frac{i^{\frac{1}{2}}\eta}{\sqrt{\alpha}} + \frac{(\tilde{M}-n)(\tilde{M}-n-\frac{2}{3})}{2}\left(\frac{i^{\frac{1}{2}}\eta}{\sqrt{\alpha}}\right)^2 + \cdots\right) \end{aligned}$$

and

$$ds = i y^{\frac{1}{2\alpha}} d\xi = i y^{\frac{1}{2\alpha}} \frac{i^{\frac{1}{2}}}{\sqrt{\alpha}} \left(1 + \frac{i^{\frac{1}{2}}\eta}{3\sqrt{\alpha}} - \frac{i\eta^2}{24\alpha} + \cdots\right) d\eta,$$

hence

$$\left(\sum_{n=0}^{\infty} p_n s^{\tilde{M}-n}\right) ds = \sum_{n=0}^{\infty} p_n (i y^{\frac{1}{2\alpha}})^{\tilde{M}-n+1} i^{\frac{1}{2}} \sum_{j=0}^{\infty} q_{n,j} (i^{\frac{1}{2}}\eta)^j d\eta$$

with some real constants $q_{n,j}$.

Now take the integral of η over $[-\delta, \delta]$. The terms of odd powers of η vanish. By using the well known formula

$$\int_{-\delta}^{\delta} e^{-a\eta^2} \eta^{2h} d\eta = a^{-h-1/2}\Gamma(h+1/2) + O_{\delta,h}\left(\frac{e^{-a\delta^2}}{a}\right) \quad a > 0,$$

we find that

$$
\begin{aligned}
&\int_{\mathcal{L}_0} \left(\exp\left(-y^{\frac{1}{2\alpha}}\eta^2\right) \sum_{n=0}^{\infty} p_n s^{\tilde{M}-n} \right) ds \\
&\sim \sum_{n=0}^{\infty}\sum_{h=0}^{\infty} p_n q_{n,2h} \left(iy^{\frac{1}{2\alpha}}\right)^{\tilde{M}-n+1} i^{\frac{1}{2}+h} \int_{-\delta}^{\delta} e^{-y^{\frac{1}{2\alpha}}\eta^2}\eta^{2h} d\eta \\
&= \sum_{n=0}^{\infty}\sum_{h=0}^{\infty} p_n q_{n,2h} \left(iy^{\frac{1}{2\alpha}}\right)^{\tilde{M}-n+1} i^{\frac{1}{2}+h} \left(y^{-\frac{h+\frac{1}{2}}{2\alpha}}\Gamma(h+1/2) + O\left(\frac{e^{-y^{\frac{1}{2\alpha}}\delta^2}}{y^{\frac{1}{2\alpha}}} \right) \right) \\
&= \sum_{n=0}^{\infty}\sum_{h=0}^{\infty} p_n q_{n,2h} \Gamma(h+1/2) i^{\tilde{M}-n+\frac{3}{2}+h} y^{\frac{\tilde{M}-n-h+\frac{1}{2}}{2\alpha}} + O\left(y^{\frac{\tilde{M}}{2\alpha}} e^{-y^{\frac{1}{2\alpha}}\delta^2} \right).
\end{aligned}
$$

Hence for any $m > 0$, we get

$$
J_0 = C_0 e^{-2\alpha i y^{\frac{1}{2\alpha}}} e^{\pi i(\alpha r+\mu-\frac{1}{2})} \sum_{k=0}^{m} c'_k i^{\tilde{M}-k+\frac{3}{2}} y^{\frac{\tilde{M}-k+\frac{1}{2}}{2\alpha}} + O\left(y^{\frac{\tilde{M}-m-1/2}{2\alpha}} \right),
$$

where we put

$$
c'_k = \sum_{n+h=k} p_n q_{n,2h} \Gamma(h+1/2)(-1)^h.
$$

Next we consider the integral on $\mathcal{L}_2$. Let

$$
J_2 = \int_{\mathcal{L}_2} \frac{\Delta_2(s)}{\Delta_1(r-s)} s^{\omega} F(s) x^{-s} ds.
$$

From the definition of $\mathcal{L}_2$, we have

$$
|J_2| \le \int_{v_2}^{\infty} \left| \frac{\Delta_2(s)}{\Delta_1(r-s)} s^{\omega} F(s) x^{-s} \right| y^{\frac{1}{2\alpha}} dv
$$

with $s = y^{\frac{1}{2\alpha}} w$, $(w = u_2 + iv)$. By using (3.12), the above integrand is bounded as

$$
\ll x^{-u_2} y^{\frac{\tilde{M}+1}{2\alpha}} |w|^{\tilde{M}} |\exp(f(s))|.
$$

Since

$$
\begin{aligned}
f(y^{\frac{1}{2\alpha}} w) &= 2\alpha y^{\frac{1}{2\alpha}} w \left(\frac{1}{2\alpha}\log y + \log w \right) - 2\alpha y^{\frac{1}{2\alpha}} w - \pi i \alpha y^{\frac{1}{2\alpha}} w - y^{\frac{1}{2\alpha}} w \log y \\
&= 2\alpha y^{\frac{1}{2\alpha}} \left(w\log w - w - \frac{\pi i}{2} w \right),
\end{aligned}
$$

we find that

$$
|J_2| \ll x^{-u_2} y^{\frac{\tilde{M}+1}{2\alpha}} \max_{v_2 \le v < \infty} G(v) \int_{v_2}^{\infty} |w|^{-2} dv,
$$

where we put

$$G(v) = v^{\tilde{M}+2} \exp\left(2\alpha y^{\frac{1}{2\alpha}} \mathrm{Re}\left(w\log w - w - \frac{\pi i}{2}w\right)\right).$$

We shall see that $G(v)$ is a decreasing function. Differentiate $G(v)$ with respect to v, then we have

$$\begin{aligned}\frac{d}{dv}G(v) =& v^{\tilde{M}+1} \exp\left(2\alpha y^{\frac{1}{2\alpha}} \mathrm{Re}\left(w\log w - w - \frac{\pi i}{2}w\right)\right) \\ & \times \left(\tilde{M} + 2 - 2\alpha y^{\frac{1}{2\alpha}} v\left(\arg w - \frac{\pi}{2}\right)\right).\end{aligned}$$

Note that

$$\lim_{v\to\infty} v\left(\arg w - \frac{\pi}{2}\right) = -u_2 > 0.$$

Therefore for sufficiently large x (hence for sufficiently large y), $G(v)$ is a decreasing function. Thus $G(v)$ attains its maximal value at $v = v_2$, namely, we have

$$|J_2| \ll x^{-u_2} y^{\frac{\tilde{M}+1}{2\alpha}} \exp\left(2\alpha y^{\frac{1}{2\alpha}} \mathrm{Re}\left(w_2\left(\log w_2 - 1 - \frac{\pi i}{2}\right)\right)\right).$$

Let

$$\delta_2 = \arg w_2 > \frac{\pi}{2}.$$

Then

$$\mathrm{Re}\left(w_2\left(\log w_2 - 1 - \frac{\pi i}{2}\right)\right) = u_2(\log|w_2| - 1) - v_2\left(\delta_2 - \frac{\pi}{2}\right),$$

which is a constant smaller than 0 (since δ is also small). Consequently J_2 is exponentially decayed as $x \to \infty$.

Similarly it is shown that J_1 is also exponentially decayed as $x \to \infty$.

Collecting these formulas, we get

$$\begin{aligned}J :=& \int_{\mathcal{L}_0+\mathcal{L}_1+\mathcal{L}_2} \frac{\Delta_2(s)}{\Delta_1(r-s)} s^{\omega} F(s) x^{-s} ds \\ =& C_0 e^{-2\alpha i y^{\frac{1}{2\alpha}}} e^{\pi i(\alpha r + \mu - \frac{1}{2})} \sum_{k=0}^{m} c_k' i^{\tilde{M}-k+\frac{3}{2}} y^{\frac{\tilde{M}-k+\frac{1}{2}}{2\alpha}} + O\left(y^{\frac{\tilde{M}-m-1/2}{2\alpha}}\right)\end{aligned}$$

for any m. Noting that $\mathcal{I}(x) = J - \bar{J}$ and $y = xe^{-(\tau+\tau')}$, we conclude that

$$\begin{aligned}\mathcal{I}(x) =& 2iC_0 \sum_{k=0}^{m} c_k' y^{\frac{\tilde{M}-k+\frac{1}{2}}{2\alpha}} \sin\left(-2\alpha y^{\frac{1}{2\alpha}} + \left(\tilde{M} - k + 2\alpha r + 2\mu + \frac{1}{2}\right)\frac{\pi}{2}\right) \\ & + O\left(y^{\frac{\tilde{M}-m-\frac{1}{2}}{2\alpha}}\right)\end{aligned}$$

$$= 2iC_0 e^{-\frac{(\tau+\tau')(\tilde{M}+\frac{1}{2})}{2\alpha}} \sum_{k=0}^{m} c_k x^{\frac{\tilde{M}-k+\frac{1}{2}}{2\alpha}} \cos\left(hx^{\frac{1}{2\alpha}} - \left(M + \omega - k + \alpha r + \mu' + \mu - \frac{1}{2}\right)\frac{\pi}{2}\right)$$
$$+ O\left(x^{\frac{\tilde{M}-m-\frac{1}{2}}{2\alpha}}\right),$$

where we write

$$c_k = c'_k e^{\frac{(\tau+\tau')}{2\alpha}k}.$$

This completes the proof of Lemma 2. □

Lemma 3. *We assume* (3.7) *in Lemma 2. Let $F(s)$ be a function which satisfies the same assumptions in Lemma 2. Define the function $\mathcal{J}(x)$ by*

$$\mathcal{J}(x) = \frac{1}{2\pi i}\int_{\mathcal{C}_{a,b}} \frac{\Gamma(r-s)\Delta_2(s)}{\Gamma(r+\rho-s)\Delta_1(r-s)} F(s)x^{-s}ds,$$

with real ϱ. Then there exist constants a_l and c_l such that for any positive integer m, we have

$$\mathcal{J}(x) = \sum_{l=0}^{m} a_l x^{\frac{\tilde{M}-l+1/2}{2\alpha}} \cos(hx^{\frac{1}{2\alpha}} + c_l\pi) + O(x^{\frac{\tilde{M}-m-1/2}{2\alpha}}) + O(x^{-b}), \tag{3.13}$$

where

$$\tilde{M} = \mu' - \mu - \alpha r - \varrho + M.$$

If we take b large, we have

$$\mathcal{J}(x) \ll x^{\frac{\mu'-\mu-\alpha r-\varrho+M+1/2}{2\alpha}}. \tag{3.14}$$

Proof. The right hand side except the last error term in (3.13) are obtained by putting $\omega = -\rho$ in Lemma 2. When we deform the path of integration $\mathcal{C}_{a,b}$ to the path $\mathcal{L}_0 \cup \mathcal{L}_1 \cup \mathcal{L}_2$ and its complex conjugate, it may pass across the poles of $\Gamma(r-s)$ greater than b, but they are finite depending on m. Hence the contribution from these poles is $O(x^{-b})$. The assertion (3.14) is obtained by taking $m = 0$ in (3.13). □

We note that if $F(s) = P(s)/Q(s)$ is a rational function of s, we can take M as $M = \deg P - \deg Q$.

Lemma 4. *Let $x > 0$ and $\varrho > \min\{2\alpha a - r\alpha + \mu' - \mu, -1\}$. Define the function $f_\varrho(x)$ by*

$$f_\varrho(x) = \frac{1}{2\pi i}\int_{\mathcal{C}_{a,b}} G_\varrho(s)x^{r+\varrho-s}ds$$

where

$$G_\varrho(s) = \frac{\Gamma(r-s)\Delta_2(s)}{\Gamma(r+\varrho+1-s)\Delta_1(r-s)}.$$

Then

$$\frac{d}{dx}f_{\varrho+1}(x) = f_\varrho(x)$$

and for any non-negative integer m, we have an asymptotic expansion

$$f_\varrho(x) = \sum_{l=0}^{m} \kappa_l x^{\theta_\varrho - \frac{l}{2\alpha}} \cos\left(hx^{\frac{1}{2\alpha}} + c_l\pi\right) + O\left(x^{\theta_\varrho - \frac{m+1}{2\alpha}} + x^{r+\varrho-b}\right), \tag{3.15}$$

where

$$c_l = -\frac{1}{2}\left(\mu + \mu' + r\alpha + \varrho + \frac{1}{2}\right) + \frac{l}{2}$$

and κ_l are constants. In particular,

$$\kappa_0 = (2\pi)^{\frac{N'-N}{2}} \left(\frac{2\alpha}{h}\right)^\varrho \sqrt{\frac{2}{h\pi}}\, e^{\nu'-\nu-\frac{(\tau'+\tau)(\mu'-\mu)}{2\alpha}+\frac{r}{2}(\tau'-\tau)},$$

and

$$\kappa_1 = \kappa_0 \frac{2\alpha}{h}\left(-\frac{1}{2\alpha}\left(\frac{(\mu'-\mu-\alpha r-\varrho-1)^2}{2} - \frac{1}{24}\right) + B\right),$$

where we put

$$B = \frac{1}{2}\left(\sum_{h=1}^{N'} \frac{B_2(\beta'_h)}{\alpha'_h} + \sum_{j=1}^{N} \frac{B_2(\alpha_j r + \beta_j)}{a_j} - B_2(r) + B_2(r+\varrho+1)\right).$$

Proof. (3.15) is obtained from Lemma 3. The other assertions are calculated explicitly by using Lemma 2. □

4 The Tong-type identity for the error term

In this section we shall prove the Tong-type identity for the integral (or multiple integral) of the error term $E_\varrho(y)$, which is a generalization of Theorem 1 in Tong [48] for the error term in the Piltz divisor problem. The main result of this section is

Theorem 4. *Let $x \geq 1$, $L \geq 0$. Suppose that*

$$\varrho + k > 2\alpha\sigma_b^* - \alpha r - \frac{1}{2} + \mu' - \mu.$$

Then for $x + ky > 0$ we have

$$\int_0^y \cdots \int_0^y E_\varrho(x + y_1 + \cdots + y_k)dy_1 \cdots dy_k$$

$$= \sum_{\mu_n \le L} \frac{b(n)}{\mu_n^{r+\varrho}} \int_0^y \cdots \int_0^y f_\varrho\left((x+y_1+\cdots+y_k)\mu_n\right) dy_1 \cdots dy_k$$
$$+ \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \sum_{\mu_n > L} \frac{b(n)}{\mu_n^{r+\varrho+k}} f_{\varrho+k}\left((x+jy)\mu_n\right).$$

Remark 7. Cao [1] and Fomenko [11] used Tong's idea in [48] to study the asymmetric many-dimensional problem and mean value theorems for automorphic L-functions, respectively.

Proof. We assume first that $\varrho \ge 0$ and $\varrho + k > \alpha(2c-r) + \mu' - \mu$. Then by Perron's formula (see, Ivić [19, (A.8) and (A.4)]) we have

$$A_\varrho(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\varphi(s)x^{s+\varrho}}{\Gamma(s+\varrho+1)} ds,$$

where c is the constant chosen in Section 2. By the residue theorem and (2.5) we have

$$E_\varrho(x) = \frac{1}{2\pi i} \int_{\mathcal{C}_{c,r-b}} \frac{\Gamma(s)\varphi(s)x^{s+\varrho}}{\Gamma(s+\varrho+1)} ds,$$

and

$$\int_{\lambda_1}^{x} E_\varrho(u) du = \frac{1}{2\pi i} \int_{\mathcal{C}_{c,r-b}} \frac{\Gamma(s)\varphi(s)x^{s+\varrho+1}}{\Gamma(s+\varrho+2)} ds + c_1$$

with some constant c_1. By induction we deduce

$$\begin{aligned}\int_0^y E_\varrho(x+y_1)dy_1 &= \int_{\lambda_1}^{x+y} E_\varrho(u)du - \int_{\lambda_1}^{x} E_\varrho(u)du \\ &= \frac{1}{2\pi i} \int_{\mathcal{C}_{c,r-b}} \frac{\Gamma(s)\varphi(s)\left((x+y)^{s+\varrho+1} - x^{s+\varrho+1}\right)}{\Gamma(s+\varrho+2)} ds,\end{aligned}$$

and

(4.1)

$$\int_0^y \cdots \int_0^y E_\varrho(x+y_1+\cdots+y_k)dy_1\cdots dy_k$$
$$= \frac{1}{2\pi i} \int_0^y \cdots \int_0^y dy_2 \cdots dy_k \int_{\mathcal{C}_{c,r-b}} \frac{\Gamma(s)\varphi(s)\left((x+y+\sum_{j=2}^{k} y_j)^{s+\varrho+1} - (x+\sum_{j=2}^{k} y_j)^{s+\varrho+1}\right)}{\Gamma(s+\varrho+2)} ds$$

$$= \frac{1}{2\pi i}\int_{\mathcal{C}_{c,r-b}} \frac{\Gamma(s)\varphi(s)}{\Gamma(s+\varrho+2)}\int_0^y\cdots\int_0^y\left(\Big(x+y+\sum_{j=2}^k y_j\Big)^{s+\varrho+1}-\Big(x+\sum_{j=2}^k y_j\Big)^{s+\varrho+1}\right)dy_2\cdots dy_k$$

$$= \sum_{j=0}^k(-1)^{k-j}\binom{k}{j}\frac{1}{2\pi i}\int_{\mathcal{C}_{c,r-b}}\frac{\Gamma(s)\varphi(s)(x+jy)^{s+\varrho+k}}{\Gamma(s+\varrho+k+1)}ds.$$

Furthermore, under the condition $\varrho+k>\alpha(2c-r)+\mu'-\mu$. we can change the path of integration in (4.1) to $\mathcal{C}_{r-c,r-b}$, and by using the functional equation along with a change of variable from s to $r-s$, we have

$$\frac{1}{2\pi i}\int_{\mathcal{C}_{c,r-b}}\frac{\Gamma(s)\varphi(s)u^{s+\varrho+k}}{\Gamma(s+\varrho+k+1)}ds = \frac{1}{2\pi i}\int_{\mathcal{C}_{c,b}} G_{\varrho+k}(s)\psi(s)u^{r+\varrho+k-s}ds \quad (u>0).$$

Since $b>c>\sigma_b^*$, $\psi(s)$ is expressed as a Dirichlet series. Exchanging the order of summation and integration, we get

$$\sum_{n=1}^\infty\frac{b(n)}{\mu_n^{r+\varrho+k}}\frac{1}{2\pi i}\int_{\mathcal{C}_{c,b}}G_{\varrho+k}(s)(u\mu_n)^{r+\varrho+k-s}ds = \sum_{n=1}^\infty\frac{b(n)}{\mu_n^{r+\varrho+k}}f_{\varrho+k}(u\mu_n) \tag{4.2}$$

for $a<c$. Since $f_{\varrho+k}(u\mu_n)\ll(u\mu_n)^{\theta_{\varrho+k}}$, the series (4.2) converges absolutely if

$$\varrho+k>2\alpha\sigma_b^*-\alpha r-\frac{1}{2}+\mu'-\mu. \tag{4.3}$$

Thus under this assumption, the exchange of the order of summation and integration is justified and the equality

$$\begin{aligned}\int_0^y\cdots\int_0^y E_\varrho\left(x+\sum_{j=1}^k y_j\right)dy_1\cdots dy_k \\ = \sum_{j=0}^k(-1)^{k-j}\binom{k}{j}\sum_{n=1}^\infty\frac{b(n)}{\mu_n^{r+\varrho+k}}f_{\varrho+k}\left((x+jy)\mu_n\right)\end{aligned} \tag{4.4}$$

is valid under (4.3).

Finally we note that if $\min\{u,u+y\}>0$, then for any non-negative integer j we have

$$\begin{aligned}\int_0^y f_{\varrho+j}\left((u+y_1)\mu_n\right)dy_1 &= \int_0^y dy_1\frac{1}{2\pi i}\int_{\mathcal{C}_{a,b}}G_{\varrho+j}(s)\left((u+y_1)\mu_n\right)^{r+\varrho+j-s}ds \\ &= \frac{1}{2\pi i\mu_n}\int_{\mathcal{C}_{a,b}}\frac{\Gamma(r-s)\Delta_2(s)}{\Gamma(r+\varrho+j+2-s)\Delta_1(r-s)} \\ &\qquad\times\Big(\left((u+y_1)\mu_n\right)^{r+\varrho+j+1-s}-(u\mu_n)^{r+\varrho+j+1-s}\Big)ds \\ &= \frac{1}{\mu_n}\left(f_{\varrho+j+1}(u+y)-f_{\varrho+j+1}(u)\right).\end{aligned} \tag{4.5}$$

By induction we get

(4.6)

$$\int_0^y \cdots \int_0^y f_\varrho\left(\left(x+\sum_{j=1}^k y_j\right)\mu_n\right) dy_1\cdots dy_k = \frac{1}{\mu_n^k}\sum_{j=0}^k (-1)^{k-j}\binom{k}{j} f_{\varrho+k}\left((x+jy)\mu_n\right).$$

Theorem 4 follows from (4.4) and (4.6) at once. □

5 Large and small values of $E_\varrho(y)$ in short intervals

In this section we shall study the large and small values of the error term $E_\varrho(y)$ for y in some "short" interval. We always suppose that $\alpha > \frac{1}{2}$.

We first state some conditions.

(A1) $b(n) \ll_\varepsilon {\mu_n}^{\ell_1+\varepsilon}$ for some $\ell_1 \geq 0$ and $b(n) \gg 1$ for infinitely many n.

(A2) $n^\hbar \ll \mu_n \ll n^\hbar$ for some $0 < \hbar$, and $\mu_m^c - \mu_n^c \gg m^{\hbar c} - n^{\hbar c}$ for $0 < c \leq \frac{1}{\hbar}$ and $m > n$.

(A3) $a(n) \ll_\varepsilon {\lambda_n}^{\ell+\varepsilon}$ for some $\ell \geq 0$ and $a(n) \gg 1$ for infinitely many n.

(A4) $a(n) \geq 0$ for all $n \in \mathbb{N}$ and $a(n) \gg 1$ for infinitely many n.

Theorem 5. *Let $\lambda \geq 1$ be a fixed real number. Assume that the conditions (A1) and (A2) hold. Then there exists a constant $B > 0$ such that for $x \geq x_0$ and $Bx^{1-\frac{1}{2\alpha}} \leq U \leq x$ we have*

$$\int_x^{x+U} |E_\varrho(y)|^\lambda dy \gg x^{\lambda\theta_\varrho} U, \quad \int_1^x |E_\varrho(y)|^\lambda dy \gg x^{1+\lambda\theta_\varrho}, \tag{5.1}$$

where θ_ϱ is defined by (3.6). In particular

$$\int_x^{x+U} |E_\varrho(y)| dy \gg x^{\theta_\varrho} U, \quad \int_1^x |E_\varrho(y)| dy \gg x^{1+\theta_\varrho}. \tag{5.2}$$

Theorem 6. *Under the conditions (A1) and (A2), there exist two positive constants B and B' such that for $x \geq x_0$, we have*

$$\max_{x\leq y\leq x+Bx^{1-\frac{1}{2\alpha}}} \pm E_\varrho(y) > B' x^{\theta_\varrho}.$$

Furthermore we have:

(i) *If the conditions (A1), (A2) and (A3) hold, then there exists a point $x^* \in [x, x+Bx^{1-\frac{1}{2\alpha}}]$ with $|E(x^*)| \ll x^{\ell+\varepsilon}$ and $E(y)$ is continuous at x^*.*

(ii) *If the conditions (A1), (A2) and (A4) hold, then for any t with $|t| < B'x^{\theta_0}$ there exists at least a point $x^* \in [x, x+Bx^{1-\frac{1}{2\alpha}}]$ such that $E(x^*) = t$ and $E(y)$ is continuous at x^*.*

Remark 8. Tong [48] first proved Theorem 6 for the error term in the summatory function of the the general Piltz divisor problem $d_\ell(n)$ for any $\ell \geq 2$. A. Ivić [20] studied the general case for $\Delta_1(s) = \Delta_2(s)$. Our result here provides a new proof of Ivić's result.

Remark 9. When $\varrho > 0$, $E_\varrho(y)$ is a continuous function of y. Then for any t with $|t| < B'x^{\theta_\varrho}$ there exists at least a point $x^* \in [x, x+Bx^{1-\frac{1}{2\alpha}}]$ such that $E(x^*) = t$ and $E(y)$ is continuous at x^*.

Remark 10. For $E^*_\varrho(y)$, we have similar results. Suppose that $\{\lambda_n\}(n \geq 1)$ satisfy (A2*): $n^{\hbar^*} \ll \lambda_n \ll n^{\hbar^*}$ for some $0 < \hbar^*$, and $\mu_m^{c^*} - \mu_n^{c^*} \gg m^{\hbar^* c^*} - n^{\hbar^* c^*}$ for $0 < c^* \leq \frac{1}{\hbar^*}$ and $m > n$.

So we have the following results:

(1) If (A3) and (A2*) hold, then Theorem 5 holds for $E^*_\varrho(y)$ with θ_ϱ replaced by

$$\theta'_\varrho =: \frac{r}{2} - \frac{1}{4\alpha} + \varrho\left(1 - \frac{1}{2\alpha}\right) + \frac{\mu - \mu'}{2\alpha}.$$

(2) If (A3) and (A2*) hold, there exists two positive constants B and B' such that for $x \geq x_0$, we have

$$\max_{x \leq y \leq x + Bx^{1-\frac{1}{2\alpha}}} \pm E^*_\varrho(y) > B'x^{\theta'_\varrho}.$$

(3) If the conditions (A3), (A2*) and (A1) hold, then there exists a point $x^* \in [x, x + Bx^{1-\frac{1}{2\alpha}}]$ with $|E^*(x^*)| \ll x^{\ell_1+\varepsilon}$ and $E^*(y)$ is continuous at point x^*.

(4) If the conditions (A3), (A2*) hold, $b(n) \geq 0 (n \geq 1)$ and $b(n) \gg 1$ for infinitely many n, then for any t with $|t| < B'x^{\theta'_0}$ there exists at least a point $x^* \in [x, x + Bx^{1-\frac{1}{2\alpha}}]$ such that $E^*(x^*) = t$ and $E^*(y)$ is continuous at point x^*.

Proof of Theorem 5. We let k be a fixed large integer such that $v = r + \varrho + k - \theta_{\varrho+k} > \sigma_b^* + 1/2$. Therefore the series

$$g(t) = \sum_{n=1}^{\infty} \frac{b(n)}{\mu_n^v} \cos\left(h(t\mu_n)^{\frac{1}{2\alpha}} + c_0\pi\right)$$

is absolutely convergent. By Lemma 2 of Ivić [20], there exist two constants $B' > 0, D' > 0$ such that for $x \geq x_0$ every interval $[x, x + D'x^{1-\frac{1}{2\alpha}}]$ contains two points x_1, x_2 for which

$$g(x_1) > B', \quad g(x_2) < -B'.$$

Take $U' = c'x^{1-\frac{1}{2\alpha}}$, where c' be a constant to be specified below, and $x^* = x_j$ for $j = 1$ or 2. Applying Theorem 4 with $L = 0$, Lemma 4 and (4.5) we have

$$\int_0^{U'} dy \int_0^y \cdots \int_0^y E_\varrho(x^* + y_1 + \cdots + y_k) dy_1 \cdots dy_k$$

$$= \int_0^{U'} \left(\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \sum_{n=1}^{\infty} \frac{b(n)}{\mu_n^{r+\varrho+k}} f_{\varrho+k}\left((x^*+jy)\mu_n\right) \right) dy$$
$$= (-1)^k \kappa_0(\varrho+k) U' g(x^*)(x^*)^{\theta_{\varrho+k}} + O\left(U' x^{\theta_{\varrho+k}-\frac{1}{2\alpha}}\right)$$
$$+ \sum_{j=1}^{k} \frac{(-1)^{k-j}}{j} \binom{k}{j} \sum_{n=1}^{\infty} \frac{b(n)}{\mu_n^{r+\varrho+k+1}} \left(f_{k+\varrho+1}\left((x^*+jU')\mu_n\right) - f_{k+\varrho+1}(x^*\mu_n)\right).$$

Using Lemma 4 again we obtain

$$\left| \int_0^{U'} dy \int_0^{y} \cdots \int_0^{y} E_\varrho(x^*+y_1+\cdots+y_k) dy_1 \cdots dy_k \right| \tag{5.3}$$
$$\geq (x^*)^{\theta_{\varrho+k+1}} \left(c'\kappa_0(\varrho+k)B' + O\left(\frac{U'}{x}\right) \right.$$
$$\left. -\kappa_0(\varrho+k+1)2^{k+1}\psi_1(r+\varrho+k+1-\theta_{\varrho+k+1}) \left(1+\frac{kU'}{x}\right)^{\theta_{\varrho+k+1}} \right),$$

where $\psi_1(s) = \sum_{n=1}^{\infty} \frac{|b(n)|}{\mu_n^s}$. Now we can choose a constant c' such that

$$c' > \frac{\kappa_0(\varrho+k+1)2^{k+2}\psi_1(r+\varrho+k+1-\theta_{\varrho+k+1})}{\kappa_0(\varrho+k)B'}. \tag{5.4}$$

Combining (5.3) and (5.4) we have for some constant $B_0 > 0$ as $x > x_0$

$$B_0 x^{\theta_{\varrho+k+1}} \leq \left| \int_0^{U'} dy \int_0^{y} \cdots \int_0^{y} E_\varrho(x^*+y_1+\cdots+y_k) dy_1 \cdots dy_k \right| \tag{5.5}$$
$$\leq \int_0^{U'} dy \int_0^{y} \cdots \int_0^{y} |E_\varrho(x^*+y_1+\cdots+y_k)| dy_1 \cdots dy_k$$
$$\leq \int_0^{U'} dy \int_0^{U} |E_\varrho(x+y_1)| dy_1 \int_0^{y} \cdots \int_0^{y} dy_2 \cdots dy_k$$
$$\leq x^{k(1-\frac{1}{2\alpha})} \int_x^{x+U} |E_\varrho(y)| dy,$$

here $U = (k+2)U'$.This proves the first assertion in (5.2).

Next, we have

$$\int_1^x |E_\varrho(y)| dy \geq \sum_{j=0}^{\frac{1}{2B}x^{\frac{1}{2\alpha}}} \int_{\frac{x}{2}+jBx^{1-\frac{1}{2\alpha}}}^{\frac{x}{2}+(j+1)Bx^{1-\frac{1}{2\alpha}}} |E_\varrho(y)| dy \gg x^{\frac{1}{2\alpha}} \left(x^{\theta_\varrho} x^{1-\frac{1}{2\alpha}} \right) = x^{1+\theta_\varrho},$$

and this completes the proof of the second assertion in (5.2).

By Hölder's inequality and (5.2) we immediately get (5.1).

Proof of Theorem 6. Similarly to (5.5) we see that for $j = 1, 2$

$$x^{\theta_{\varrho+k+1}} \ll \max \pm \int_0^{U'} dy \int_0^y \cdots \int_0^y E_\varrho(x_j + y_1 + \cdots + y_k) dy_1 \cdots dy_k$$
$$\ll \max_{x \le y \le x+U} \pm E_\varrho(y) \times \int_0^{U'} dy \int_0^y \cdots \int_0^y dy_2 \cdots dy_k.$$

Hence for $x > x_0$

$$\max_{x \le y \le x+U} \pm E_\varrho(y) \gg x^{\theta_\varrho}. \tag{5.6}$$

When $\varrho = 0$, for every interval $[x, x + 2U]$, it follows from (5.6) that there exists two points $y', y'' \in [x, x + 2U]$ such that

$$E(y') > 0, E(y'') < 0, \quad y' < y''.$$

If the condition (A4) holds, then $Q(y)$, which is the main term of the summatory function $a(n)$, is of the form (2.6) with $\varrho = 0$, always positive and increasing. And the function $E(y)$ is a continuous and strictly decreasing in each interval $(\lambda_n, \lambda_{n+1})$. Moreover, $E(y)$ has a jump up to $a(n)/2$ at the point $\lambda_n \ \ (n = 1, 2, \cdots)$, i.e. $E(\lambda_n) = E(\lambda_n - 0) + a(n)/2$. Hence there exists at least a zero point x^* of $E(x)$ in the interval (y', y'') with $E(x)$ is continuous at point x^*.

Otherwise, we may suppose that for any $x \in (y', y'')$ with $E(x) \neq 0$. Without loss of generality, we assume that for all $\lambda_n \in (y', y'')$ with $E(\lambda_n + 0)E(\lambda_{n+1} - 0) > 0$. Hence we can find at least a $\lambda_n \in (y', y'')$ such that $E(\lambda_n - 0) \times E(\lambda_n + 0) < 0$, in this time we can choose $x^* = \frac{\lambda_n + \lambda_{n+1}}{2}$. This completes the proof of Theorem 6.

6 The truncated Tong-type formula for an integral involving the error term

Suppose $x > 10$ is a large parameter. For any integrable function $g(y)$, we define

$$\tilde{y} := y + \frac{1}{x}(y_1 + \cdots + y_k),$$
$$\int_{\mathbf{E}_k} g(\tilde{y}) d\mathrm{Y}_k := \int_0^1 \cdots \int_0^1 g(\tilde{y}) dy_1 \cdots dy_k.$$

The integral $\int_{\mathbf{E}_k} E(\tilde{y}) d\mathrm{Y}_k$ is very important in Tong's theory. The aim of this section is to give its truncated expression under some suitable conditions.

Recall the definition of $A^*(y)$. Let $\hat{Q}(y)$ denote the sum of residues over all poles of the function $\psi(s) y^s s^{-1}$ except the pole $s = 0$. And define the error term $\hat{E}(y)$ by

$$\hat{E}(y) := A^*(y) - \hat{Q}(y).$$

Since $\psi(s)$ doesn't have poles when $\sigma \le \sigma^*$ by assumption, we have immediately that

$$Q^*(y) = \hat{Q}(y) + \psi(0), \quad E^*(y) = \hat{E}(y) - \psi(0).$$

Let $y \ge 1$, $\frac{1}{2} \le M \le N < \infty$, λ be a real number, and define

$$I(\lambda, M, N, y) = 2\pi i \int_M^N u^\lambda \hat{E}(u) \exp\left(-ih(uy)^{\frac{1}{2\alpha}}\right) du. \tag{6.1}$$

We assume in this section that

(B1) There exists a constant $1 - \sigma_b^* \le \omega_0 < 1$ such that $|b(n)| \ll \mu_n^{\sigma_b^* - 1 + \omega_0}$.

(B2) $\sum_{\mu_n \le y} |b(n)| \ll y^{\sigma_b^*} \log^A y, \ y \ge 2$.

(B3) There exists a constant $0 \le \omega_1 \le 1$ such that

$$\int_1^T |\hat{E}(u)| du \ll T^{\sigma_b^* + \omega_1}.$$

Throughout this section we let k to be a fixed positive integer such that

$$k > \max\left\{2\alpha\sigma_b^* - \alpha r - \frac{1}{2} - \varrho + \mu' - \mu, \alpha r\right\}. \tag{6.2}$$

Furthermore we assume that

$$\sigma_b^* + \omega_1 - r - \frac{1}{4\alpha} - \frac{\varrho}{2\alpha} + \frac{\mu' - \mu}{2\alpha} - 1 < 0. \tag{6.3}$$

We note that (6.2) and (6.3) implies

$$\sigma_b^* + \omega_1 - \frac{r}{2} - \frac{1}{4\alpha} - 1 - \frac{\varrho}{2\alpha} + \frac{\mu' - \mu}{2\alpha} - \frac{k}{\alpha} < 0. \tag{6.4}$$

Let $1 \le x \le y \le (1+\delta)x$, $N = [x^{4\alpha - 1 - \varepsilon}]$ and $J = \left[(4\alpha^2 r + 4\alpha)\varepsilon^{-1}\right]$, where $\delta > 0$ is a fixed small positive constant. Without loss of generality, we suppose $N \ne \mu_n$ for any n. Furthermore let

$$\lambda_0 := \theta_\varrho + \frac{1}{2\alpha} - r - \varrho - 1 = -\frac{r}{2} + \frac{1}{4\alpha} - \frac{\varrho}{2\alpha} + \frac{\mu' - \mu}{2\alpha} - 1.$$

From Theorem 6 we can find a real number M in every subinterval $[t, t + Bt^{1 - \frac{1}{2\alpha}}]$ in $[1, \sqrt{N}]$ such that $M \ne \mu_n$ for $n = 1, 2, \ldots$ and $\hat{E}(M) = 0$ if $b(n) \ge 0$ for all n or $\hat{E}(M) \ll M^{\sigma_b^* - 1 + \omega_0}$ otherwise.

The truncated Tong-type formula can be stated as

Theorem 7. *Assume the conditions (A2), (B1), (B2) and (B3) hold. Let $M \in [1, \sqrt{N}]$ be chosen as above. Then we have*

$$\int_{\mathbf{E}_k} E_\varrho(\tilde{y}) d\mathrm{Y}_k = \sum_{j=1}^{6} R_j(y; M), \tag{6.5}$$

where

$$R_1(y; M) = \kappa_0 y^{\theta_\varrho} \sum_{\mu_n \le M} \frac{b(n)}{\mu_n^{r+\varrho-\theta_\varrho}} \cos(h(y\mu_n)^{\frac{1}{2\alpha}} + c_0\pi), \tag{6.6}$$

$$R_2(y; M) = y^{\theta_\varrho + \frac{1}{2\alpha}} \mathrm{Re}\left\{c_{00} I\left(\lambda_0, M, N, y\right)\right\}, \tag{6.7}$$

$$R_3(y; M) = \sum_{\substack{l=0 \\ l+m\neq 0}}^{J} \sum_{m=0}^{J} \mathrm{Re}\left\{c_{lm} I\left(\lambda_0 + \frac{l-m}{2\alpha}, M, N, y\right)\right\} x^{-l} y^{-l+\theta_\varrho + \frac{1}{2\alpha} + \frac{l-m}{2\alpha}}, \tag{6.8}$$

$$\begin{aligned} R_4(y; M) = &\sum_{j=0}^{k} \sum_{m=0}^{k} \mathrm{Re}\left\{c'_{jm} I\left(\lambda_0 - \frac{k+m}{2\alpha}, N, \infty, y + \frac{j}{x}\right)\right\} \\ &\times x^k \left(y + \frac{j}{x}\right)^{k+\theta_\varrho + \frac{1}{2\alpha} - \frac{k+m}{2\alpha}}, \end{aligned} \tag{6.9}$$

$$\begin{aligned} R_5(y; M) \ll &x^{\theta_\varrho - \frac{1}{2\alpha}} M^{\max(\sigma_b^* - \frac{r}{2} - \frac{3}{4\alpha} - \frac{\varrho}{2\alpha} + \frac{\mu'-\mu}{2\alpha}, 0)} \log^A M \\ &+ x^{-2+\theta_\varrho + \frac{1}{2\alpha}} M^{\max(\sigma_b^* - \frac{r}{2} + \frac{1}{4\alpha} - \frac{\varrho}{2\alpha} + \frac{\mu'-\mu}{2\alpha}, 0)} \log^A M \\ &+ x^{\theta_\varrho - \frac{r}{2}} M^{\sigma_b^* + \omega_1 - r - \frac{1}{4\alpha} - \frac{\varrho}{2\alpha} + \frac{\mu'-\mu}{2\alpha} - 1} \\ &+ x^{(4\alpha-1)(\sigma_b^* + \omega_1) - 2k + r - \varrho + 2(\mu'-\mu) + \frac{2k}{\alpha} - 2\alpha r - 4\alpha}, \end{aligned} \tag{6.10}$$

$$R_6(y; M) \ll \begin{cases} 0, & \text{if } b(n) \ge 0 \ (n \ge 1), \\ x^{\theta_\varrho} M^{\sigma_b^* - 1 + \omega_0 - \frac{r}{2} - \frac{1}{4\alpha} - \frac{\varrho}{2\alpha} + \frac{\mu'-\mu}{2\alpha}}, & \text{if the condition (B1) holds.} \end{cases} \tag{6.11}$$

Here κ_0, c_0 and c_{00}, c_{lm}, c'_{jm} are certain real and complex constants, respectively.

6.1 Lemmas for generalized Bessel functions

Before proving Theorem 7 we shall prepare some preliminary assertions for the so-called generalized Bessel functions.

By induction one easily verifies ($r - s \notin -\mathbb{N}$)

$$\int_{\mathbf{E}_k} \tilde{y}^{r-s} d\mathrm{Y}_k = x^k \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} \frac{(y + lx^{-1})^{k+r-s}}{(1+r-s)(2+r-s)\cdots(k+r-s)} \tag{6.12}$$

For $j \geq 0$, define

$$f_{\varrho,j}(x) = \frac{1}{2\pi i} \int_{\mathcal{C}_{a,b}} \frac{\Gamma(r-s)}{(r+\varrho-s)^j \Gamma(r+\varrho+1-s)} \frac{\Delta_2(s)}{\Delta_1(r-s)} x^{r+\varrho-s} ds$$

and

$$g_{\varrho,j}(u) = \int_{\mathbf{E}_k} f_{\varrho,j}(\tilde{y}u) d\mathrm{Y}_{\mathrm{k}}.$$

By definition $f_\varrho(x) = f_{\varrho,0}(x)$. We note that $f_{\varrho,j}(x)$ converges absolutely if

$$a < \frac{r}{2} + \frac{\varrho + j - \mu' + \mu}{2\alpha}.$$

Lemma 5. *Let $x \leq y \leq (1+\delta)x$, b is a sufficiently large positive constant. Then we have*

$$f_{\varrho,j}(x) \ll x^{\theta_\varrho - \frac{j}{2\alpha}}, \tag{6.13}$$

and

$$g_{\varrho,j}(u) \ll x^{2k} (ux)^{\theta_\varrho - \frac{k+j}{2\alpha}}. \tag{6.14}$$

Proof. The first assertion (6.13) is obtained immediately by replacing ϱ by $\varrho+1$ and taking $M = -j$ in (3.14). (M is the parameter in the definition of $F(s)$, see (3.9).)

To prove (6.14), substituting the definitions of $f_{\varrho,j}(x)$ into $g_{\varrho,j}(u)$ and using (6.12), we get

$$\begin{aligned} g_{\varrho,j}(u) = x^k & \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} \left(y + \frac{l}{x}\right)^k \\ & \times \frac{1}{2\pi i} \int_{\mathcal{C}_{a,b}} \frac{\Gamma(r-s)}{(r+\varrho-s)^j \Gamma(r+\varrho+k+1-s)} \frac{\Delta_2(s)}{\Delta_1(r-s)} \left(\left(y + \frac{l}{x}\right) u\right)^{r+\varrho-s} ds. \end{aligned}$$

Replacing ϱ by $\lambda = \varrho + k + 1$ and taking $M = -j$ in (3.14), we get

$$g_{\varrho,j}(u) \ll x^{2k} (ux)^{\theta_\varrho - \frac{k+j}{2\alpha}}.$$

□

We also need the asymptotic behaviour of the function $\Phi_\varrho(y)$ defined by

$$\Phi_\varrho(y) = \frac{d}{du} \left(\frac{f_\varrho(yu)}{u^{r+\varrho}} \right).$$

Lemma 6. *Let l be a fixed positive integer and $x \leq y \leq (1+\delta)x$. Then there are constants b'_{lm} and b''_{jm} such that for any positive integer H, we have*

$$\frac{d^l}{dy^l} \Phi_\varrho(y) = \sum_{m=0}^{H} \mathrm{Re} \left\{ b'_{lm} \frac{(uy)^{\theta_\varrho + \frac{1}{2\alpha} + \frac{l-m}{2\alpha}}}{y^l u^{r+\varrho+1}} e^{-ih(uy)^{\frac{1}{2\alpha}}} + O\left(\frac{(uy)^{\theta_\varrho + \frac{l-H}{2\alpha}}}{y^l u^{r+\varrho+1}} \right) \right\} \tag{6.15}$$

$$+ O\left(y^{r+\varrho-b-l}u^{-1-b}\right),$$

and

$$(6.16)\qquad \int_{\mathbf{E}_k}\Phi_\varrho(\tilde{y})d\mathrm{Y}_\mathrm{k} = \sum_{j=0}^{k}\left\{\sum_{m=0}^{H}\mathrm{Re}\left(b''_{jm}\frac{x^k(y+\frac{j}{x})^{k+\theta_\varrho+\frac{1}{2\alpha}-\frac{k+m}{2\alpha}}}{u^{r+\varrho+1-\frac{1}{2\alpha}-\theta_\varrho+\frac{k+m}{2\alpha}}}e^{-ih((y+\frac{j}{x})u)^{\frac{1}{2\alpha}}}\right)\right.$$
$$\left. +O\left(\frac{x^{2k+\theta_\varrho+\frac{1}{2\alpha}-\frac{k+H+1}{2\alpha}}}{u^{r+\varrho+1-\frac{1}{2\alpha}-\theta_\varrho+\frac{k+H+1}{2\alpha}}}\right)\right\} + O\left(x^{2k+r+\varrho-b}u^{-1-b}\right).$$

Proof. From the definition of $f_\varrho(yu)$, we have

$$(6.17)\qquad \Phi_\varrho(y) = \frac{1}{2\pi i}\int_{\mathcal{C}_{a,b}}\frac{\Gamma(r-s)\Delta_2(s)(-s)}{\Gamma(r+\varrho+1-s)\Delta_1(r-s)}y^{r+\varrho-s}u^{-s-1}ds,$$

hence

$$\frac{d^l}{dy^l}\Phi_\varrho(y) = \frac{1}{y^l u^{r+\varrho+1}}\frac{(-1)^{l-1}}{2\pi i}\int_{\mathcal{C}_{a,b}}\frac{\Gamma(r-s)\Delta_2(s)}{\Gamma(r+\varrho+1-s)\Delta_2(r-s)}$$
$$\times s(s-r-\varrho)(s-r-\varrho+1)\cdots(s-r-\varrho+l-1)(yu)^{r+\varrho-s}ds.$$

Replacing ϱ by $\varrho+1$ and taking $M=l+1$ in the formula (3.13), we find that

$$\frac{d^l}{dy^l}\Phi_\varrho(y) \sim \frac{1}{y^l u^{r+\varrho+1}}\Big\{\sum_{m=0}^{\infty}b_{lm}(yu)^{\theta_\varrho+\frac{1}{2\alpha}+\frac{l-m}{2\alpha}}\cos(h(yu)^{\frac{1}{2\alpha}}+c_m\pi)$$
$$+O((yu)^{r+\varrho-b})\Big\},$$

which proves the assertion (6.15).

Next we prove (6.16). From (6.17) and (6.12) we have

(6.18)
$$\int_{\mathbf{E}_k}\Phi_\varrho(\tilde{y})d\mathrm{Y}_\mathrm{k} = \frac{1}{2\pi i}\int_{\mathcal{C}_{a,b}}\frac{\Gamma(r-s)\Delta_2(s)(-s)}{\Gamma(r+\varrho+1-s)\Delta_1(r-s)}u^{-s-1}\int_{\mathbf{E}_k}\tilde{y}^{r+\varrho-s}d\mathrm{Y}_\mathrm{k}ds$$
$$= \frac{x^k}{u^{r+\varrho+1}}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}\left(y+\frac{j}{k}\right)^k$$
$$\times\frac{1}{2\pi i}\int_{\mathcal{C}_{a,b}}\frac{\Gamma(r-s)\Delta_2(s)(-s)}{\Gamma(r+\varrho+k+1-s)\Delta_1(r-s)}\left(\left(y+\frac{j}{x}\right)u\right)^{r+\varrho-s}ds.$$

Replacing ϱ by $\varrho+k+1$ and taking $M=1$ in (3.13), we find that

$$\frac{1}{2\pi i}\int_{\mathcal{C}_{a,b}}\frac{\Gamma(r-s)\Delta_2(s)(-s)}{\Gamma(r+\varrho+k+1-s)\Delta_1(r-s)}\left(\left(y+\frac{j}{x}\right)u\right)^{r+\varrho-s}ds$$

$$\sim \sum_{m=0}^{\infty} b'_{jm} \left(\left(y + \frac{j}{x} \right) u \right)^{\theta_\varrho + \frac{1}{2\alpha} - \frac{k+m}{2\alpha}} \cos \left(h \left(\left(y + \frac{j}{x} \right) u \right)^{\frac{1}{2\alpha}} + c_m \pi \right)$$
$$+ O\left((yu)^{r+\varrho-b} \right)$$

with certain real constants b'_{jm}. Substituting this expression to (6.18), we get (6.16). $\square$

6.2 Proof of Theorem 7

Now we turn to the proof of Theorem 7. Let $M \in [1, \sqrt{N}]$ be a real number as is chosen before, namely. $M \neq \mu_n$ for $n = 1, 2, \ldots$ and $\hat{E}(M) = 0$ if $b(n) \geq 0$ for all n or $\hat{E}(M) \ll M^{\sigma_b^* - 1 + \omega_0}$ otherwise. We always suppose that there are infinitely many n for which $b(n) \gg 1$.

Recall the Tong-type identity:

$$I := \int_{\mathbf{E}_k} E_\varrho \left(y + \frac{\sum_{j=1}^k y_j}{x} \right) d\mathrm{Y}_k = \sum_{n=1}^{\infty} \frac{b(n)}{\mu_n^{r+\varrho}} \int_{\mathbf{E}_k} f_\varrho \left(\left(y + \frac{\sum_{j=1}^k y_j}{x} \right) \mu_n \right) d\mathrm{Y}_k.$$

In the right hand side we divide the sum over n into two parts, namely

$$I = \sum_{\mu_n \leq M} + \sum_{\mu_n > M} =: S_1(M) + S_2(M).$$

We first treat $S_1(M)$. Let $F(\tilde{y}) = f_\varrho(\tilde{y}u)$. By Taylor's formula we have

$$F(\tilde{y}) = F(y) + F'(y_1)(\tilde{y} - y)$$

with some $y \leq y_1 \leq \tilde{y}$. Then from the definition $f_\varrho(x)$ and (3.15) of Lemma 4 we have an asymptotic expansion

$$F(y) \sim \sum_{l=0}^{\infty} \kappa_l (yu)^{\theta_\varrho - \frac{l}{2\alpha}} \cos(h(yu)^{\frac{1}{2\alpha}} + c_l \pi) + O((yu)^{r+\varrho-b}) \tag{6.19}$$

and the upper bound

$$\ll (yu)^{\theta_\varrho}$$

if we take b large. Furthermore since $F'(y_1) = u f_{\varrho-1}(y_1 u)$, we have

$$|F'(y_1)(\tilde{y} - y)| \ll \frac{u}{x} (xu)^{\theta_{\varrho-1}}$$
$$= x^{-2} (xu)^{\theta_\varrho + \frac{1}{2\alpha}}.$$

Take out the term corresponding to $l = 0$ in (6.19), we get

$$S_1(M) = \sum_{\mu_n \leq M} \frac{b(n)}{\mu_n^{r+\varrho}} \Big\{ \kappa_0 (y\mu_n)^{\theta_\varrho} \cos(h(y\mu_n)^{\frac{1}{2\alpha}} + c_0 \pi) + O((y\mu_n)^{\theta_\varrho - \frac{1}{2\alpha}})$$

$$+O((y\mu_n)^{r+\varrho-b}) + O\left(x^{-2}(x\mu_n)^{\theta_\varrho+\frac{1}{2\alpha}}\right)\Big\}$$
$$=: R_1(y;M) + V_1 + V_2 + V_3.$$

The error term V_2 can be negligible by taking b large. On the hand, by (B2), we have

$$V_1 \ll x^{\theta_\varrho - \frac{1}{2\alpha}} \sum_{\mu_n \le M} |b(n)|\mu_n^{-r-\varrho+\theta_\varrho-\frac{1}{2\alpha}}$$
$$\ll x^{\theta_\varrho-\frac{1}{2\alpha}} M^{\max(\sigma_b^* - \frac{r}{2} - \frac{3}{4\alpha} - \frac{\varrho}{2\alpha} + \frac{\mu'-\mu}{2\alpha}, 0)} \log^A M. \tag{6.20}$$

Similarly we have

$$V_3 \ll x^{-2+\theta_\varrho+\frac{1}{2\alpha}} M^{\max(\sigma_b^* - \frac{r}{2} + \frac{1}{4\alpha} - \frac{\varrho}{2\alpha} + \frac{\mu'-\mu}{2\alpha}, 0)} \log^A M. \tag{6.21}$$

Since $M \ne \mu_n$ for all n, it is easy to see that if $K(u)$ is any continuously differentiable function such that

$$K(u)Q^*(u) \to 0, \int_u^\infty Q^*(t)K'(t)dt \to 0 \ (u \to \infty),$$

then

$$\int_M^\infty K(u) d \sum_{\mu_n \le u} b(n) = \int_M^\infty K(u) dA^*(u).$$

So for $S_2(M)$ we can write

$$\begin{aligned}
S_2(M) &= \int_M^\infty \left(\frac{1}{u^{r+\varrho}} \int_{E_k} f_\varrho(\tilde{y}u) d\mathrm{Y}_{\mathrm{k}}\right) d\sum_{\mu_n \le u} b(n) \\
&= \int_M^\infty \left(\frac{1}{u^{r+\varrho}} \int_{E_k} f_\varrho(\tilde{y}u) d\mathrm{Y}_{\mathrm{k}}\right) dA^*(u) \\
&= \int_M^\infty \left(\frac{1}{u^{r+\varrho}} \int_{E_k} f_\varrho(\tilde{y}u) d\mathrm{Y}_{\mathrm{k}}\right) d\hat{Q}(u) + \int_M^\infty \left(\frac{1}{u^{r+\varrho}} \int_{E_k} f_\varrho(\tilde{y}u) d\mathrm{Y}_{\mathrm{k}}\right) d\hat{E}(u) \\
&= \int_M^\infty \frac{\hat{Q}'(u)}{u^{r+\varrho}} \int_{E_k} f_\varrho(\tilde{y}u) d\mathrm{Y}_{\mathrm{k}} du + \frac{\hat{E}(u)}{u^{r+\varrho}} \int_{E_k} f_\varrho(\tilde{y}u) d\mathrm{Y}_{\mathrm{k}} \Big|_{u=M}^\infty \\
&\quad - \int_M^\infty \hat{E}(u) du \int_{\mathbf{E}_k} \frac{d}{du}\left(\frac{f_\varrho(\tilde{y}u)}{u^{r+\varrho}}\right) d\mathrm{Y}_k \\
&=: S_{21}(M) + S_{22}(M) - S_{23}(M).
\end{aligned}$$

We shall consider $S_{21}(M)$ first. From the definition of $f_{\varrho,j}(x)$, one easily checks that

$$\frac{d}{du} f_{\varrho,j+1}(yu) = \frac{1}{u} f_{\varrho,j}(yu) \quad (1 \le j \le k+1).$$

Let $w_j(u)$ be the functions defined by

$$w_1(u) = \hat{Q}'(u)u^{1-r-\varrho}$$

and

$$w_{j+1}(u) = w_j'(u)u \ (j \geq 1).$$

Then by repeated integration by parts and the definition of $g_{\varrho,j}$ we have

$$\begin{aligned}(6.22)\qquad \int_M^{M_1} \frac{\hat{Q}'(u)}{u^{r+\varrho}} \int_{E_k} f_\varrho(\tilde{y}u) d\mathrm{Y_k} du &= \sum_{j=1}^{k+1} (-1)^{j+1} w_j(u) g_{\varrho,j}(u)\Big|_M^{M_1} \\ &\quad + (-1)^{k+1} \int_M^{M_1} w_{k+1}'(u) g_{\varrho,k+1}(u) du.\end{aligned}$$

Recall that $\hat{Q}(u)$ is the sum of residues of of all poles of $\psi(s)x^s s^{-1}$ except 0. Let γ be a pole with order ν and γ_0 the pole with the maximal real part. Then the residue at γ has the form $\hat{Q}_\gamma(u) = u^\gamma P(\log u)$, where $P(u)$ is a polynomial of degree $\nu - 1$. Let $w_{j,\gamma}(u)$ be functions defined by the same way for $\hat{Q}_\gamma(u)$. Then we see easily

$$\begin{aligned} w_{1,\gamma}(u) &= u^{1-r-\varrho}(\gamma u^{\gamma-1} P(\log u) + u^\gamma P'(\log u)/u) \\ &\ll u^{\mathrm{Re}\,\gamma - r - \varrho} \log^{\nu-1} u. \end{aligned}$$

Similarly we have

$$w_{j,\gamma}(u) \ll u^{\mathrm{Re}\,\gamma - r - \varrho} \log^{\nu-1} u$$

for $j \geq 2$. Hence by (6.14), we have

$$w_{j,\gamma}(M_1) g_{\varrho,j}(M_1) \ll M_1^{\mathrm{Re}\,\gamma - r - \varrho} (\log M_1)^{\nu-1} x^{2k} (xM_1)^{\theta_\varrho - \frac{k+j}{2\alpha}}.$$

Since k is chosen as (6.2), the exponent of M_1 is negative, $w_{j,\gamma}(M_1) g_{\varrho,j}(M_1)$, as well as $w_j(M_1) g_{\varrho,j}(M_1)$ tend to 0 when $M_1 \to \infty$.

Next we consider the term with $u = M$. Applying (6.13) in this case, we get

$$\begin{aligned}(6.23)\qquad |w_{j,\gamma}(M) g_{\varrho,j}(M)| &\leq |w_{j,\gamma}(M)| \int_{E_k} |f_{\varrho,j}(\tilde{y}u)| d\mathrm{Y_k} \\ &\ll M^{\mathrm{Re}\,\gamma - r - \varrho} (\log M)^{\nu-1} (xM)^{\theta_\varrho - \frac{1}{2\alpha}} \\ &= x^{\theta_\varrho - \frac{j}{2\alpha}} M^{\mathrm{Re}\,\gamma - \frac{r}{2} - \frac{1+2j}{4\alpha} - \frac{\varrho}{2\alpha} + \frac{\mu'-\mu}{2\alpha}} (\log M)^{\nu-1}.\end{aligned}$$

For the last integral of (6.22) which corresponds to γ, (6.13) implies that

$$\begin{aligned}(6.24)\qquad \int_M^{M_1} w_{k+1,\gamma}'(u) g_{\varrho,k+1}(u) du &\ll \int_M^{M_1} u^{\mathrm{Re}\,\gamma - r - \varrho} (\log u)^{\nu-1} (xu)^{\theta_\varrho - \frac{k+1}{2\alpha}} du \\ &= x^{\theta_\varrho - \frac{k+1}{2\alpha}} M^{\mathrm{Re}\,\gamma - \frac{r}{2} - \frac{1}{4\alpha} - \frac{\varrho}{2} + \frac{\mu'-\mu}{2\alpha} - \frac{k+1}{2\alpha}} (\log M)^{\nu-1}.\end{aligned}$$

From (6.23) and (6.24) we get the estimate from the pole γ. It is easily seen that the term $j = 1$ coming from γ_0 gives the maximal estimate, namely we have

$$S_{21}(M) \ll x^{\theta_\varrho - \frac{1}{2\alpha}} M^{\mathrm{Re}\,\gamma_0 - \frac{r}{2} - \frac{3}{4\alpha} - \frac{\varrho}{2\alpha} + \frac{\mu' - \mu}{2\alpha}} (\log M)^{A'}$$

with some integer A'. Now we note that $\mathrm{Re}\,\gamma_0 \le \sigma_b^*$, hence $S_{21}(M)$ is absorbed into the right hand side of (6.20).

Next we consider $S_{22}(M)$. By the trivial bound $\hat{E}(x) \ll x^{\sigma_b^* + \varepsilon}$ and (6.14), we see that

$$\frac{\hat{E}(M_1)}{M_1^{r+\varrho}} \int_{\mathbf{E}_k} f_\varrho(\tilde{y} M_1) d\mathrm{Y_k} \ll M_1^{\sigma_b^* - r - \varrho + \varepsilon} x^{2k} (y M_1)^{\theta_\varrho - \frac{k}{2\alpha}}$$

The exponent of M_1 is negative by (6.2), hence this term vanishes when $M_1 \to \infty$.

Consider the term $\frac{\hat{E}(M)}{M^{r+\varrho}} \int_{E_k} f_\varrho(\tilde{y} M) dY$. If we assume that $b(n) \ge 0 (n \ge 1)$, then from Theorem 6 (ii) we can find M such that $\hat{E}(M) = 0$, hence this term becomes zero. If we assume the condition (B1), there exists M such that $\hat{E}(M) \ll M^{\sigma_b^* - 1 + \omega_0}$. Hence by (6.13)

$$\begin{aligned} S_{22}(M) &\ll x^{\theta_\varrho} M^{\sigma_b^* - 1 + \omega_0 - (r+\varrho) + \theta_\varrho} \\ &= x^{\theta_\varrho} M^{\sigma_b^* - 1 + \omega_0 - \frac{r}{2} - \frac{1}{4\alpha} - \frac{\varrho}{2\alpha} + \frac{\mu' - \mu}{2\alpha}}. \end{aligned}$$

We call the above estimate of $S_{22}(M)$ as $R_6(M)$.

Finally we consider $S_{23}(M)$. Divide the integral as

$$S_{23}(M) = \left(\int_M^N + \int_N^\infty \right) \hat{E}(u) \int_{\mathbf{E}_k} \frac{d}{du} \left(\frac{f_\varrho(\tilde{y} u)}{u^{r+\varrho}} \right) d\mathrm{Y_k} du =: S_{231} + S_{232}.$$

To treat S_{231}, we substitute Taylor's formula for $\Phi_\varrho(\tilde{y}) = \frac{d}{du} \left(\frac{f_\varrho(\tilde{y} u)}{u^{r+\varrho}} \right)$:

$$\Phi_\varrho(\tilde{y}) = \sum_{l=0}^{J} \left(\frac{d^l}{dy^l} \Phi_\varrho(y) \right) \frac{(\tilde{y} - y)^l}{l!} + \frac{d^{J+1}}{dv^{J+1}} \Phi_\varrho(v) \Big|_{v = v_0} \frac{(\tilde{y} - y)^{J+1}}{(J+1)!}, \tag{6.25}$$

where $v_0 = y + \theta(\tilde{y} - y)$ with some $0 \le \theta \le 1$. Let V_4 be the contribution in S_{231} from the last error term of (6.25). From (6.16) and $v_0 \sim y \sim x$, we have

$$\begin{aligned} V_4 &= \int_M^N \hat{E}(u) du \int_{\mathbf{E}_k} \frac{d^{J+1}}{dv^{J+1}} \Phi_\varrho(v) \Big|_{v = v_0} \frac{(\tilde{y} - y)^{J+1}}{(J+1)!} d\mathrm{Y}_k \\ &\ll \int_M^N |\hat{E}(u)| \frac{(xu)^{\theta_\varrho + \frac{1}{2\alpha} + \frac{J+1}{2\alpha}}}{x^{J+1} u^{r+\varrho+1}} \frac{1}{x^{J+1}} du \\ &\ll \int_M^N |\hat{E}(u)| \frac{(xu)^{\theta_\varrho - \frac{r}{2}}}{u^{r+\varrho+1}} (xu)^{\frac{r}{2} + \frac{1}{2\alpha}} \left(\frac{(xu)^{\frac{1}{2\alpha}}}{x^2} \right)^{J+1} du. \end{aligned}$$

Since $u \le N = [x^{4\alpha-1-\varepsilon}]$ and $J = [(4\alpha^2 r + 4\alpha)\varepsilon^{-1}]$, it follows that

$$\begin{aligned}(ux)^{\frac{r}{2}+\frac{1}{2\alpha}}\left(\frac{(ux)^{\frac{1}{2\alpha}}}{x^2}\right)^{J+1} &\le (x^{4\alpha-1-\varepsilon}x)^{\frac{r}{2}+\frac{1}{2\alpha}}\left(\frac{(x^{4\alpha-1-\varepsilon}x)^{\frac{1}{2\alpha}}}{x^2}\right)^{J+1}\\ &\le x^{-\varepsilon(\frac{r}{2}+\frac{1}{2\alpha})} \le 1.\end{aligned}$$

Therefore

$$V_4 \ll x^{\theta_\varrho-\frac{r}{2}}\int_M^N |\hat{E}(u)| u^{-r-\frac{1}{4\alpha}-\frac{\varrho}{2\alpha}+\frac{\mu'-\mu}{2\alpha}-1}du.$$

Now by the assumption (B3) we get the upper bound of V_4:

$$V_4 \ll x^{\theta_\varrho-\frac{r}{2}}M^{\sigma_b^*+\omega_1-r-\frac{1}{4a}-\frac{\varrho}{2\alpha}+\frac{\mu'-\mu}{2\alpha}-1}. \tag{6.26}$$

Next we treat the first J-terms of the Taylor expansion of $\Phi_\varrho(\tilde{y})$. Their contribution to S_{231} is

$$\begin{aligned}&\sum_{l=0}^{J}\frac{x^{-l}}{l!}\int_M^N \hat{E}(u)\frac{d^l}{dy^l}\Phi_\varrho(y)du\int_{\mathbf{E}_k}(y_1+\cdots+y_k)^l d\mathrm{Y}_k\\ &=\sum_{l=0}^{J}c_l^* x^{-l}\int_M^N \hat{E}(u)\frac{d^l}{dy^l}\Phi_\varrho(y)du\\ &=\sum_{l=0}^{J}\sum_{m=0}^{J}\mathrm{Re}\left\{c_l^* b'_{lm}\int_M^N \hat{E}(u)\frac{(uy)^{\theta_\varrho+\frac{1}{2\alpha}+\frac{l-m}{2\alpha}}}{(xy)^l u^{r+\varrho+1}}e^{-ih(uy)^{\frac{1}{2\alpha}}}du\right\}\\ &\quad+O\left(\sum_{l=0}^{J}\int_M^N|\hat{E}(u)|\frac{(ux)^{\theta_\varrho+\frac{l-J}{2\alpha}}}{x^{2l}u^{r+\varrho+1}}du\right)+O\left(\sum_{l=0}^{J}x^{-l+r+\varrho-b-1}\int_M^N|\hat{E}(u)|u^{-1-b}du\right).\end{aligned}$$

The last O-term can be removed by taking b large. In the first O-term, the integrand can be transformed into

$$|\hat{E}(u)|\frac{(ux)^{\theta_\varrho-\frac{J}{2\alpha}}}{u^{r+\varrho+1}}\left(\frac{(ux)^{\frac{1}{2\alpha}}}{x^2}\right)^l.$$

Since $\frac{(ux)^{\frac{1}{2\alpha}}}{x^2} \le 1$, this O-term is bounded by

$$\ll \sum_{l=0}^{J}\int_M^N|\hat{E}(u)|\frac{(ux)^{\theta_\varrho-\frac{J}{2\alpha}}}{u^{r+\varrho+1}}du \ll x^{\theta_\varrho-\frac{J}{2\alpha}}M^{\sigma_b^*+\omega_1-\frac{r}{2}-\frac{1}{4\alpha}-\frac{\varrho}{2\alpha}+\frac{\mu'-\mu}{2\alpha}-\frac{J}{2\alpha}-1},$$

which is contained in the right hand side of (6.26). Combining these formulas we obtain that

$$S_{231} = \sum_{l=0}^{J}\sum_{m=0}^{J}\mathrm{Re}\left\{c_l^* b'_{lm}\int_M^N \hat{E}(u)\frac{(uy)^{\theta_\varrho+\frac{1}{2\alpha}+\frac{l-m}{2\alpha}}}{(xy)^l u^{r+\varrho+1}}e^{-ih(uy)^{\frac{1}{2\alpha}}}du\right\}$$

$$+O\left(x^{\theta_\varrho-\frac{r}{2}}M^{\sigma_b^*+\omega_1-r-\frac{1}{4a}-\frac{\varrho}{2\alpha}+\frac{\mu'-\mu}{2\alpha}-1}\right).$$

As for the double sum over l and m, let $R_2(y;M)$ denote the term corresponding to $l=m=0$ and $R_3(y;M)$ the term corresponding to $l+m>0$. With the notation of the function $I(\lambda,M,N,y)$, we can express these terms as

$$R_2(y;M)=y^{\theta_\varrho+\frac{1}{2\alpha}}\mathrm{Re}\left\{c_{00}I\left(\theta_\varrho+\frac{1}{2\alpha}-r-\varrho-1,M,N,y\right)\right\},$$

$$R_3(y;M)=\sum_{\substack{l=0\\l+m\neq0}}^{J}\sum_{m=0}^{J}\mathrm{Re}\left\{c_{lm}I\left(\theta_\varrho+\frac{1}{2\alpha}+\frac{l-m}{2\alpha}-r-\varrho-1,M,N,y\right)\right\}x^{-l}y^{\theta_\varrho+\frac{1}{2\alpha}-l+\frac{l-m}{2\alpha}}.$$

Finally we consider S_{232}. From (6.16) with $H=k$, we have

$$\int_{\mathbf{E}_k}\Phi_\varrho(\tilde y)d\mathrm{Y_k}=\sum_{j=0}^{k}\left\{\sum_{m=0}^{k}\mathrm{Re}\left(b''_{jm}\frac{x^k(y+\frac{j}{x})^{k+\theta_\varrho+\frac{1}{2\alpha}-\frac{k+m}{2\alpha}}}{u^{r+\varrho+1-\frac{1}{2\alpha}-\theta_\varrho+\frac{k+m}{2\alpha}}}e^{-ih((y+\frac{j}{x})u)^{\frac{1}{2\alpha}}}\right)\right.$$
$$\left.+O\left(\frac{x^{2k+\theta_\varrho+\frac{1}{2\alpha}-\frac{2k+1}{2\alpha}}}{u^{r+\varrho+1-\frac{1}{2\alpha}-\theta_\varrho+\frac{2k+1}{2\alpha}}}\right)\right\}+O\left(x^{2k+r+\varrho-b}u^{-1-b}\right).$$

Substituting this expression to the definition of S_{232} and noting (6.4), the contribution from the second O-term becomes

$$\ll\int_N^\infty|\hat E(u)|\frac{x^{2k+\theta_\varrho+\frac{k}{\alpha}}}{u^{\frac{r}{2}+\frac{1}{4\alpha}+1+\frac{\varrho}{2\alpha}-\frac{\mu'-\mu}{2\alpha}+\frac{k}{\alpha}}}du$$
$$\ll x^{2k+\theta_\varrho+\frac{k}{\alpha}}N^{\sigma_b^*+\omega_1-(\frac{r}{2}+\frac{1}{4\alpha}+1+\frac{\varrho}{2\alpha}-\frac{\mu'-\mu}{2\alpha}+\frac{k}{\alpha})},$$
$$\ll x^{(4\alpha-1)(\sigma_b^*+\omega_1)-2k+r-\varrho+2(\mu'-\mu)+\frac{2k}{\alpha}-2\alpha r-4\alpha}.$$

The contribution from the last O-term is also negligible by taking b large. Combining these, we deduce that

$$S_{232}=\sum_{j=0}^{k}\sum_{m=0}^{k}\mathrm{Re}\left\{c'_{jm}\int_N^\infty\hat E(u)\frac{x^k(y+\frac{j}{x})^{k+\theta_\varrho+\frac{1}{2\alpha}-\frac{k+m}{2\alpha}}}{u^{r+\varrho+1-\frac{1}{2\alpha}-\theta_\varrho+\frac{k+m}{2\alpha}}}e^{-ih((y+\frac{j}{x})u)^{\frac{1}{2\alpha}}}du\right\}$$
$$+O\left(x^{(4\alpha-1)(\sigma_b^*+\omega_1)-2k+r-\varrho+2(\mu'-\mu)+\frac{2k}{\alpha}-2\alpha r-4\alpha}\right). \tag{6.27}$$

The term of sum over j and m, which we denote by $R_4(y;M)$, can be expressed as

$$R_4(y;M)=x^k\sum_{j=0}^{k}\sum_{m=0}^{k}\mathrm{Re}\left\{c'_{lm}\left(y+\frac{j}{x}\right)^{k+\theta_\varrho+\frac{1}{2\alpha}-\frac{k+m}{2\alpha}}\right.$$
$$\left.\times I\left(-r-\varrho-1+\frac{1}{2\alpha}+\theta_\varrho-\frac{k+m}{2\alpha},N,\infty,y+\frac{j}{x}\right)\right\}$$

We call the sum of remaining terms (6.20), (6.21), (6.26) and (6.27) as $R_5(y;M)$. This completes the proof of Theorem 7. □

7 Some estimates for the weighted integral of the error term

In this section we shall give several estimates involving the exponential integral $I(\lambda, M, N, y)$ defined by (6.1), which is closely related to the mean square bound (2.7) and important in this paper. For convenience we use the notation $\int_{(\sigma)} f(s)ds = \int_{\sigma-i\infty}^{\sigma+i\infty} f(s)ds$.

We first give the following preliminary lemma.

Lemma 7. *Let $x \geq 1, \frac{1}{2} \leq P \leq P_1 < P_2 \leq (1+\delta)P$. We have*

$$\int_{P_1}^{P_2} u^{\lambda} e^{\pm i(t \log u - ux)} du \ll x^{-\frac{1}{2}} P^{\lambda+\frac{1}{2}} (1+|t|)^{-\varepsilon} (Px)^{\varepsilon}. \tag{7.1}$$

Proof. Let $f(u) = t\log u - ux$. Clearly

$$f'(u) = \frac{t}{u} - x, \quad f''(u) = -\frac{t}{u^2}.$$

First we suppose $(1-\delta)Px \leq t \leq (1+2\delta)Px$, we apply the second mean value theorem and the second derivative test to get

$$\begin{aligned} \int_{P_1}^{P_2} u^{\lambda} e^{\pm i(t \log u - ux)} du &\ll P^{\lambda} \max_{P_1 \leq P_3 \leq P_4 \leq P_2} \left| \int_{P_3}^{P_4} e^{\pm i(t \log u - ux)} du \right| \\ &\ll P^{\lambda} \max_{P_1 \leq u \leq P_2} \sqrt{\frac{u^2}{t}} \ll x^{-\frac{1}{2}} P^{\lambda+\frac{1}{2}}, \end{aligned} \tag{7.2}$$

from which the assertion of the lemma follows in this case.

If $(1+2\delta)Px < t < \infty$ or $-\infty < t < (1-\delta)Px$, by the first derivative test we get

$$\begin{aligned} \int_{P_1}^{P_2} u^{\lambda} e^{\pm i(t \log u - ux)} du &\ll P^{\lambda} \max_{P_1 \leq P_3 < P_4 \leq P_2} \left\{ \left| \frac{t}{P_3} - x \right|^{-1}, \left| \frac{t}{P_4} - x \right|^{-1} \right\} \\ &\ll P^{\lambda} x^{-1} (1+|t|)^{-\varepsilon} (Px)^{\varepsilon} \ (\text{or} \quad \ll P^{\lambda} x^{-1}). \end{aligned} \tag{7.3}$$

The assertion of the lemma can be checked easily by the above two cases. $\square$

Lemma 8. *Let $M < N \leq x^A$ (A a fixed positive number), w be a real number and $0 < \mu_n \leq \frac{M}{2}$. Then we have*

(7.4)
$$\int_{x}^{(1+\delta)x} I(\lambda, M, N, y) y^{w} \cos\left(h(\mu_n y)^{\frac{1}{2\alpha}} + c\right) dy \ll x^{w+1-\frac{3}{4\alpha}+\varepsilon'} \max_{M \leq P \leq N} P^{\lambda+\sigma^*+1-\frac{3}{4\alpha}}.$$

Proof. From (2.7) we easily get (by using Cauchy's inequality)

$$\int_{-T}^{T} |\psi(\sigma^* + it)|^2 dt \ll T^{1+\varepsilon}, \tag{7.5}$$

$$\int_{-T}^{T} |\psi(\sigma^* + it)|\, dt \ll \sqrt{2T \int_{-T}^{T} |\psi(\sigma^* + it)|^2\, dt} \ll T^{1+\frac{\varepsilon}{2}},$$

and (by using integration by parts)

(7.6)
$$\int_0^\infty \frac{|\psi(\sigma^* \pm it)|}{(1+|t|)^{1+\varepsilon}} dt = \frac{\int_0^t |\psi(\sigma^* \pm iu)|\, du}{(1+|t|)^{1+\varepsilon}}\Bigg|_{t=0}^{\infty} + (1+\varepsilon)\int_0^\infty \frac{\int_0^t |\psi(\sigma^* \pm iu)|\, du}{(1+|t|)^{2+\varepsilon}} dt \ll 1.$$

By Perron's formula and the residue theorem we have

$$\begin{aligned}\hat{E}(u) &= \lim_{T\to\infty} \frac{1}{2\pi i} \int_{\sigma_b^*+\varepsilon-iT}^{\sigma_b^*+\varepsilon+iT} \psi(s) \frac{u^s}{s} ds - \hat{Q}(u) \\ &= \lim_{T\to\infty} \frac{1}{2\pi i} \int_{\sigma^*-iT}^{\sigma^*+iT} \psi(s) \frac{u^s}{s} ds,\end{aligned}$$

since $\psi(s)s^{-1} \to 0$ uniformly in the strip $\sigma^* \le \operatorname{Re} s \le \sigma_b^*$ as $t \to \pm\infty$ (also see page 357 in Ivić [19]). Let $\frac{1}{2} \le P < Q \le (1+\delta)P$, then from (6.1)

$$\begin{aligned}I(\lambda, P, Q, y) &= \int_P^Q u^\lambda e^{-ih(uy)^{\frac{1}{2\alpha}}} du \lim_{T\to\infty} \int_{\sigma^*-iT}^{\sigma^*+iT} \psi(s) \frac{u^s}{s} ds \\ &= \lim_{T\to\infty} \int_{\sigma^*-iT}^{\sigma^*+iT} \frac{\psi(s)}{s} ds \int_P^Q u^{\lambda+s} e^{-ih(uy)^{\frac{1}{2\alpha}}} du \\ &= \int_{\sigma^*-i\infty}^{\sigma^*+i\infty} \frac{\psi(s)}{s} ds \int_P^Q u^{\lambda+\sigma^*+it} e^{-ih(uy)^{\frac{1}{2\alpha}}} du, \quad (s = \sigma^* + it),\end{aligned} \tag{7.7}$$

where the inversion of order of two integrations is justified by (7.1) and (7.6).

Assume $0 < \mu_n \le \frac{P}{2}$, we get

$$\begin{aligned}&\int_x^{(1+\delta)x} I(\lambda, P, Q, y) y^w e^{\pm h(\mu_n y)^{\frac{1}{2\alpha}}} dy \\ &= \int_x^{(1+\delta)x} y^w e^{\pm ih(\mu_n y)^{\frac{1}{2\alpha}}} dy \int_{\sigma^*-i\infty}^{\sigma^*+i\infty} \frac{\psi(s)}{s} ds \int_P^Q u^{\lambda+\sigma^*+it} e^{-ih(uy)^{\frac{1}{2\alpha}}} du \\ &= \int_{\sigma^*-i\infty}^{\sigma^*+i\infty} \frac{\psi(s)}{s} ds \int_P^Q \int_x^{(1+\delta)x} y^w u^{\lambda+\sigma^*+it} e^{i(\pm h(\mu_n y)^{\frac{1}{2\alpha}} - h(uy)^{\frac{1}{2\alpha}})} dy du \\ &=: \int_{\sigma^*-i\infty}^{\sigma^*+i\infty} \frac{\psi(s)}{s} I_1(s) ds,\end{aligned} \tag{7.8}$$

where the inversion of order of the integrations is justified by (7.1) and (7.6). Set $u^{\frac{1}{2\alpha}} = U, U_0 = P^{\frac{1}{2\alpha}}, U_1 = Q^{\frac{1}{2\alpha}}$, $hy^{\frac{1}{2\alpha}} = Y, X_0 = hx^{\frac{1}{2\alpha}}$ and $X_1 = h\left((1+\delta)x\right)^{\frac{1}{2\alpha}}$, hence

$$I_1(s) = \frac{(2\alpha)^2}{h^{2\alpha(w+1)}} \int_{U_0}^{U_1} U^{2\alpha(\lambda+1+\sigma^*+it)-1} dU \int_{X_0}^{X_1} Y^{2\alpha(w+1)-1} e^{i(\pm\mu_n^{\frac{1}{2\alpha}} - U)Y} dY. \tag{7.9}$$

We write $\eta = 2\alpha(w+1)-1$. Applying integration by parts we get

$$\begin{aligned}
&\int_{X_0}^{X_1} Y^{\eta} e^{i(\pm\mu_n^{\frac{1}{2\alpha}}-U)Y}\,dY \\
&= \frac{1}{(\pm\mu_n^{\frac{1}{2\alpha}}-U)i}\left(Y^{\eta} e^{i(\pm\mu_n^{\frac{1}{2\alpha}}-U)Y}\Big|_{X_0}^{X_1} - \eta\int_{X_0}^{X_1} Y^{\eta-1} e^{i(\pm\mu_n^{\frac{1}{2\alpha}}-U)Y}\,dY\right).
\end{aligned}$$

Substituting the above expression into (7.9) we obtain

$$I_1(s) \ll X_0^{\eta} \max_{X_0\le Y\le X_1}\left|\int_{U_0}^{U_1} \frac{U^{2\alpha(\lambda+1+\sigma^*)-1}}{\pm\mu_n^{\frac{1}{2\alpha}}-U} e^{i(2\alpha t\log U-UY)}\,dU\right|.$$

Now we apply the second mean value theorem and (7.1) to get

$$\begin{aligned}
I_1(s) &\ll \frac{X_0^{\eta}}{U_0\mp\mu_n^{\frac{1}{2\alpha}}} \max_{X_0\le Y\le X_1, U_0\le U^*\le U_1}\left|\int_{U_0}^{U^*} U^{2\alpha(\lambda+1+\sigma^*)-1} e^{i(2\alpha t\log U-UY)}\,dU\right| \\
&\ll \frac{X_0^{\eta}}{U_0} X_0^{-\frac{1}{2}} U_0^{2\alpha(\lambda+1+\sigma^*)-\frac{1}{2}} (X_0U_0)^{\varepsilon}(1+|t|)^{-\varepsilon} \\
&\ll x^{w+1-\frac{3}{4\alpha}} P^{\lambda+1+\sigma^*-\frac{3}{4\alpha}} (Px)^{\varepsilon}(1+|t|)^{-\varepsilon}.
\end{aligned}$$

We substitute the above estimate into (7.8) and then use (7.6) to get

$$\begin{aligned}
\int_x^{(1+\delta)x} I(\lambda,P,Q,y) y^w e^{\pm ih(\mu_n y)^{\frac{1}{2\alpha}}}\,dy &= \int_{\sigma^*-i\infty}^{\sigma^*+i\infty} \frac{\psi(s)}{s} I_1(s)\,ds \\
&\ll x^{w+1-\frac{3}{4\alpha}} P^{\lambda+1+\sigma^*-\frac{3}{4\alpha}} (Px)^{\varepsilon} \int_{-\infty}^{\infty} \frac{|\psi(\sigma^*+it)|}{(1+|t|)^{1+\varepsilon}}\,dt \\
&\ll x^{w+1-\frac{3}{4\alpha}} P^{\lambda+1+\sigma^*-\frac{3}{4\alpha}} (Px)^{\varepsilon}.
\end{aligned}$$

Now applying a simple splitting argument to the interval $[M,N]$ would suffice to complete the proof of the lemma. ∎

Lemma 9. *Let $2(\lambda+\sigma^*)\neq -1$, $M<N\le x^A$ (A a fixed positive number) and $\delta>0$ with $(1+\delta)^{\frac{1}{\alpha}}-1<\frac{1}{4}$. Then we have*

$$\int_x^{(1+\delta)x} |I(\lambda,M,N,y)|^2\,dy \ll x^{1-\frac{1}{\alpha}+\varepsilon'} \max_{M\le P\le N} P^{2(\lambda+\sigma^*+1)-\frac{1}{\alpha}}. \tag{7.10}$$

Proof. From (7.7) we obtain

$$\int_x^{(1+\delta)x} |I(\lambda,P,Q,y)|^2\,dy \tag{7.11}$$

$$
\begin{aligned}
&= \int_x^{(1+\delta)x} dy \left(\int_{(\sigma^*)} \frac{\psi(s)}{s} ds \int_P^Q u^{\lambda+\sigma^*} e^{i\left(t\log u - h(uy)^{\frac{1}{2\alpha}}\right)} du \right) \\
&\quad \times \left(\int_{(\sigma^*)} \frac{\psi(\bar{s}_1)}{\bar{s}_1} d\bar{s}_1 \int_P^Q v^{\lambda+\sigma^*} e^{i\left(-t_1\log v + h(vy)^{\frac{1}{2\alpha}}\right)} dv \right) \\
&= \int_{(\sigma^*)} \int_{(\sigma^*)} \frac{\psi(s)\psi(\bar{s}_1)}{s\bar{s}_1} ds d\bar{s}_1 \times I(s,s_1),
\end{aligned}
$$

where the inversion of order of the integrations is justified by (7.1) and (7.6), and we use notation $s = \sigma^* + it$, $s_1 = \sigma^* + it_1$ and

$$
I(s,s_1) := \int_x^{(1+\delta)x} dy \int_P^Q u^{\lambda+\sigma^*} e^{i\left(t\log u - h(uy)^{\frac{1}{2\alpha}}\right)} du \int_P^Q v^{\lambda+\sigma^*} e^{i\left(-t_1\log v + h(vy)^{\frac{1}{2\alpha}}\right)} dv.
$$

Set $h(uy)^{\frac{1}{2\alpha}} = U, h(vy)^{\frac{1}{2\alpha}} = V, P^{\frac{1}{2\alpha}} = U_0, Q^{\frac{1}{2\alpha}} = U_1, hy^{\frac{1}{2\alpha}} = Y, hx^{\frac{1}{2\alpha}} = X_0, h((1+\delta)x)^{\frac{1}{2\alpha}} = X_1$, $t_2 = 2\alpha t, t_3 = 2\alpha t_1$, $\mu = 2\alpha(\lambda+\sigma^*+1) - 1$ and $\beta = -4\alpha(\lambda+\sigma^*) - 2\alpha - 1$. We have

$$
I(s,s_1) = \frac{(2\alpha)^3}{h^{2\alpha}} \int_{X_0}^{X_1} Y^{\beta} e^{i(t_3-t_2)\log Y} G(Y;t_2,t_3) dY, \tag{7.12}
$$

where

$$
G(Y;t_2,t_3) := \int_{U_0Y}^{U_1Y} U^{\mu} e^{i(t_2\log U - U)} dU \int_{U_0Y}^{U_1Y} V^{\mu} e^{i(-t_3\log V + V)} dV.
$$

Since $P < Q \le (1+\delta)P$, so $U_0X_0 < U_1X_1 \le (1+\delta)^{\frac{1}{\alpha}} U_0X_0 := (1+\delta_1)U_0X_0$. Here note that $0 < \delta_1 = (1+\delta)^{\frac{1}{\alpha}} - 1 < \frac{1}{4}$.

To estimate $I(s,s_1)$ we need to consider the following four cases.

(Case 1). $t_2, t_3 \notin ((1-2\delta_1)U_0X_0, (1+4\delta_1)U_0X_0)$. In this case applying (7.3) to estimate the two integrals in $G(Y;t_2,t_3)$ we obtain

$$
\begin{aligned}
G(Y;t_2,t_3) &\ll \max_{X_0 \le Y \le X_1} \left| \int_{U_0Y}^{U_1Y} U^{\mu} e^{i(t_2\log U - U)} dU \right| \times \max_{X_0 \le Y \le X_1} \left| \int_{U_0Y}^{U_1Y} V^{\mu} e^{i(-t_3\log V + V)} dV \right| \\
&\ll (U_0X_0)^{2\mu} (U_0X_0)^{2\varepsilon} (1+|t_2|)^{-\varepsilon} (1+|t_3|)^{-\varepsilon}.
\end{aligned}
$$

We substitute the above estimate into (7.12) to get

$$
I(s,s_1) \ll U_0^{2\mu} X_0^{2\mu+\beta+1} (U_0X_0)^{2\varepsilon} (1+|t_2|)^{-\varepsilon} (1+|t_3|)^{-\varepsilon}. \tag{7.13}
$$

(Case 2). $t_2 \in ((1-2\delta_1)U_0X_0, (1+4\delta_1)U_0X_0)$, $t_3 \notin ((1-2\delta_1)U_0X_0, (1+4\delta_1)U_0X_0)$. If $|t_2 - t_3| \le \delta_1 U_0X_0$, then $t_2 < (1-\delta_1)U_0X_0$ or $t_2 > (1+3\delta_1)U_0X_0$. Clearly the estimate (7.13) also holds.

Now we assume $|t_2 - t_3| > \delta_1 U_0 X_0$. Applying integration by parts to the variable Y in (7.12) we get

$$
\begin{aligned}
(7.14) \qquad \frac{h^{2\alpha}}{(2\alpha)^3} I(s, s_1) &= \int_{X_0}^{X_1} Y^{\beta} e^{i(t_3 - t_2)\log Y} G(Y; t_2, t_3) dY \\
&= \frac{1}{\beta + 1 + (t_3 - t_2)i} \int_{X_0}^{X_1} G(Y; t_2, t_3) dY^{\beta + 1 + i(t_3 - t_2)} \\
&= \frac{1}{\beta + 1 + (t_3 - t_2)i} \left(\sum_{j=0}^{1} (-1)^{j+1} X_j^{\beta+1} e^{i(t_3 - t_2)\log X_j} G(X_j; t_2, t_3) \right. \\
&\quad \left. + \sum_{j=0}^{1} (-1)^j U_j^{\mu+1+it_2} I_{1j} + \sum_{j=0}^{1} (-1)^j U_j^{\mu+1-it_3} I_{2j} \right),
\end{aligned}
$$

where

$$
\begin{aligned}
I_{1j} &:= \int_{X_0}^{X_1} Y^{\beta+\mu+1} e^{i(t_3 \log Y - U_j Y)} dY \int_{U_0 Y}^{U_1 Y} V^{\mu} e^{i(-t_3 \log V + V)} dV \\
I_{2j} &:= \int_{X_0}^{X_1} Y^{\beta+\mu+1} e^{i(-t_2 \log Y + U_j Y)} dY \int_{U_0 Y}^{U_1 Y} U^{\mu} e^{i(t_2 \log U - U)} dU.
\end{aligned}
$$

Applying (7.2) to the first integral over U in $G(X_j; t_2, t_3)$, and (7.3) to the second integral over V in $G(X_j; t_2, t_3)$ respectively, we obtain

$$
(7.15) \qquad G(X_j; t_2, t_3) \ll (U_0 X_0)^{\mu + \frac{1}{2}} (U_0 X_0)^{\mu} \ll (U_0 X_0)^{2\mu + \frac{1}{2}}, \quad (j = 0, 1).
$$

Applying (7.3) to I_{1j} one has

$$
\begin{aligned}
(7.16) \qquad I_{1j} &\ll X_0^{\beta+\mu+2} \max_{X_0 \le Y \le X_1} \left| \int_{U_0 Y}^{U_1 Y} V^{\mu} e^{i(-t_3 \log V + V)} dV \right| \\
&\ll X_0^{\beta+\mu+2} (U_0 X_0)^{\mu} \ll U_0^{\mu} X_0^{\beta+2\mu+2}.
\end{aligned}
$$

To treat I_{2j} we define for $j = 0, 1$

$$
X_{2j} = \frac{t_2}{U_j} - \sqrt{\frac{X_0}{U_j}} \quad X_{3j} = \frac{t_2}{U_j} + \sqrt{\frac{X_0}{U_j}}.
$$

First we consider the case when $X_0 \le X_{2j} < X_{3j} \le X_1$, we divide the interval $[X_0, X_1]$ into three subintervals and write

$$
(7.17) \qquad I_{2j} = \int_{X_0}^{X_{2j}} + \int_{X_{2j}}^{X_{3j}} + \int_{X_{3j}}^{X_1} := I_{3j} + I_{4j} + I_{5j}.
$$

We let $\kappa = \beta + \mu + 1$ and $H(Y, t_2) := \int_{U_0 Y}^{U_1 Y} U^\mu e^{i(t_2 \log U - U)} dU$. It follows that from integration by parts

$$I_{3j} = \int_{X_0}^{X_{2j}} Y^\kappa H(Y, t_2) e^{i(-t_2 \log Y + U_j Y)} dY \tag{7.18}$$
$$= \int_{X_0}^{X_{2j}} \frac{Y^{\kappa+1}}{i(-t_2 + U_j Y)} H(Y, t_2) d e^{i(-t_2 \log Y + U_j Y)}$$
$$= I_{6j} + I_{7j} - I_{8j},$$

where*

$$I_{6j} := \frac{Y^{\kappa+1}}{i(-t_2 + U_j Y)} H(Y, t_2) e^{i(-t_2 \log Y + U_j Y)} \Big|_{X_0}^{X_{2j}},$$
$$I_{7j} := \sum_{l=0}^{1} (-1)^l U_l^{\mu+1+it_2} \int_{X_0}^{X_{2j}} \frac{Y^{\kappa+\mu+1}}{i(t_2 - U_j Y)} e^{i(U_j - U_l) Y} dY,$$
$$I_{8j} := \int_{X_0}^{X_{2j}} \left(\frac{(\kappa+1) Y^\kappa}{i(-t_2 + U_j Y)} - \frac{Y^{\kappa+1} U_j}{i(-t_2 + U_j Y)^2} \right) H(Y, t_2) e^{i(-t_2 \log Y + U_j Y)} dY.$$

By (7.2) we get

$$I_{6j} \ll X_0^{\kappa+1} \max_{X_0 \le Y \le X_{2j}} \frac{1}{|t_2 - U_j Y|} \times (U_0 X_0)^{\mu + \frac{1}{2}}$$
$$\ll X_0^{\kappa+1} \frac{1}{\sqrt{X_0 U_0}} (U_0 X_0)^{\mu+\frac{1}{2}} \ll U_0^\mu X_0^{\kappa+\mu+1}.$$

We also have

$$I_{7j} \ll U_0^{\mu+1} \int_{X_0}^{X_{2j}} \frac{X_0^{\kappa+\mu+1}}{t_2 - U_j Y} dY \ll U_0^\mu X_0^{\kappa+\mu+1} \log(U_0 X_0).$$

Applying (7.2) to the integral $H(Y, t_2)$ in I_{8j} we obtain

$$I_{8j} \ll \int_{X_0}^{X_{2j}} \left(\frac{X_0^\kappa}{t_2 - U_j Y} + \frac{X_0^{\kappa+1} U_0}{(t_2 - U_j Y)^2} \right) (U_0 X_0)^{\mu+\frac{1}{2}} dY$$
$$\ll U_0^{\mu+\frac{1}{2}} X_0^{\kappa+\mu+\frac{1}{2}} \int_{X_0}^{X_{2j}} \frac{1}{t_2 - U_j Y} dY + U_0^{\mu+\frac{3}{2}} X_0^{\kappa+\mu+\frac{3}{2}} \int_{X_0}^{X_{2j}} \frac{1}{(t_2 - U_j Y)^2} dY$$
$$\ll U_0^{\mu-\frac{1}{2}} X_0^{\kappa+\mu+\frac{1}{2}} \log(U_0 X_0) + U_0^\mu X_0^{\kappa+\mu+1} \ll U_0^\mu X_0^{\kappa+\mu+1}.$$

Combining the above three estimates and (7.18) we have

$$I_{3j} \ll U_0^\mu X_0^{\kappa+\mu+1} \log(U_0 X_0). \tag{7.19}$$

*In Tong's paper[49], page 521, he missed the term I_{8j}.

Similarly to the estimate for I_{8j} one has

$$I_{4j} \ll \int_{X_{2j}}^{X_{3j}} X_0^{\kappa}(U_0X_0)^{\mu+\frac{1}{2}}dY \ll (X_{3j}-X_{2j})U_0^{\mu+\frac{1}{2}}X_0^{\kappa+\mu+\frac{1}{2}}$$
$$\ll \sqrt{\frac{X_0}{U_0}}U_0^{\mu+\frac{1}{2}}X_0^{\kappa+\mu+\frac{1}{2}} \ll U_0^{\mu}X_0^{\kappa+\mu+1}. \tag{7.20}$$

Similarly to the estimate for I_{3j} we can prove

$$I_{5j} \ll U_0^{\mu}X_0^{\kappa+\mu+1}\log(U_0X_0). \tag{7.21}$$

Substituting (7.19), (7.20) and (7.21) into (7.17) we obtain when $X_0 \le X_{2j} < X_{3j} \le X_1$

$$I_{2j} \ll U_0^{\mu}X_0^{\kappa+\mu+1}\log(U_0X_0).$$

It remains to consider the other four cases: $X_{2j} < X_{3j} \le X_0 < X_1$, $X_{2j} \le X_0 < X_{3j} \le X_1$, $X_0 \le X_{2j} \le X_1 < X_{3j}$ and $X_0 \le X_1 \le X_{2j} < X_{3j}$. By a very similar approach to the case $X_0 \le X_{2j} < X_{3j} \le X_1$, one may get the same estimate of I_{2j} for the above four cases. For example, when $\le X_{2j} \le X_0 < X_{3j} \le X_1$ we use decomposition

$$I_{2j} = \int_{X_0}^{X_1} = \int_{X_0}^{X_{3j}} + \int_{X_{3j}}^{X_1}.$$

Hence we always have

$$I_{2j} \ll U_0^{\mu}X_0^{\beta+2\mu+2}\log(U_0X_0). \tag{7.22}$$

Since $|t_2-t_3| > \delta_1U_0X_0$ and $t_2 \in ((1-2\delta_1)U_0X_0, (1+4\delta_1)U_0X_0)$, it is easy to check

$$|t_2-t_3| > (U_0X_0)^{1-2\varepsilon}(1+|t_2|)^{-\varepsilon}(1+|t_3|)^{-\varepsilon}.$$

Substituting (7.15), (7.16) and (7.22) into (7.14), and noting the above estimate, in (Case 2) we finally obtain

$$I(s,s_1) \ll U_0^{2\mu}X_0^{2\mu+\beta+1}(U_0X_0)^{3\varepsilon}(1+|t_2|)^{-\varepsilon}(1+|t_3|)^{-\varepsilon}. \tag{7.23}$$

(Case 3). $t_2 \notin ((1-2\delta_1)U_0X_0, (1+4\delta_1)U_0X_0)$, $t_3 \in ((1-2\delta_1)U_0X_0, (1+4\delta_1)U_0X_0)$.

We can use a similar approach as (Case 2) to estimate $I(s,s_1)$ and get the same estimate as (7.23).

(Case 4). $t_2, t_3 \in ((1-2\delta_1)U_0X_0, (1+4\delta_1)U_0X_0)$.

Note that $\beta+1+i(t_3-t_2) \ne 0$. We first transform $I(s,s_1)$ into the expression (7.14). Then we use the same approach as I_{2j} in (Case 2) to treat I_{2j} and estimate I_{1j} respectively. Lastly by the second derivative test (7.2) to each integral in $G(X_j;t_2,t_3)$, we get

$$G(X_j;t_2,t_3) \ll (U_0X_0)^{\mu+\frac{1}{2}}(U_0X_0)^{\mu+\frac{1}{2}} \ll (U_0X_0)^{2\mu+1}, \quad (j=0,1).$$

Combining these estimates and (7.14) we obtain

$$I(s,s_1) \ll \frac{U_0^{2\mu+1}X_0^{\beta+2\mu+2}}{1+|t_2-t_3|}(U_0X_0)^{\varepsilon}$$

Now let $h(Px)^{\frac{1}{2\alpha}} = T$. Note that $2\mu+\beta+1 = 2\alpha-2$. Combining four estimates in the above cases and (7.11) we get

$$\begin{aligned}
(7.24)\qquad & \int_x^{(1+\delta)x} |I(\lambda,P,Q,y)|^2\,dy = \int_{(\sigma^*)}\int_{(\sigma^*)} \frac{\psi(s)\psi(\bar{s}_1)}{s\bar{s}_1} I(s,s_1)\,ds\,d\bar{s}_1 \\
& \ll x^{1-\frac{1}{\alpha}}P^{\frac{\mu}{\alpha}}T^{3\varepsilon}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{|\psi(\sigma^*+it)\psi(\sigma^*-it_1)|}{(1+|t|)(1+|t_1|)}\frac{dtdt_1}{(1+|t|)^{\varepsilon}(1+|t_1|)^{\varepsilon}} \\
& \quad + x^{1-\frac{1}{2\alpha}}P^{\frac{2\mu+1}{2\alpha}}T^{\varepsilon}\int_{(1-2\delta_1)T}^{(1+4\delta_1)T}\int_{(1-2\delta_1)T}^{(1+4\delta_1)T} \frac{|\psi(\sigma^*+it)\psi(\sigma^*-it_1)|}{tt_1}\frac{dtdt_1}{1+|t-t_1|} \\
& =: x^{1-\frac{1}{\alpha}}P^{\frac{\mu}{\alpha}}T^{3\varepsilon}W_1 + x^{1-\frac{1}{2\alpha}}P^{\frac{2\mu+1}{2\alpha}}T^{\varepsilon}W_2.
\end{aligned}$$

It follows from (7.6) that

$$(7.25)\qquad W_1 = \int_{-\infty}^{\infty}\frac{|\psi(\sigma^*+it)|}{(1+|t|)^{1+\varepsilon}}dt \times \int_{-\infty}^{\infty}\frac{|\psi(\sigma^*-it_1)|}{(1+|t_1|)^{1+\varepsilon}}dt_1 \ll 1.$$

Making the change of variables $t = u+v, t_1 = u-v$, using Cauchy's inequality and noting (7.5), it is easy to check

(7.26)

$$\begin{aligned}
W_2 & \ll T^{-2}\int_0^{2T}\int_0^{2T}\frac{|\psi(\sigma^*+it)\psi(\sigma^*-it_1)|}{1+|t-t_1|}dtdt_1 \\
& \ll T^{-2}\int_{-T}^{T}\frac{1}{1+|v|}\left(\int_0^{2T}|\psi(\sigma^*+i(u+v))\psi(\sigma^*-i(u-v))|du\right)dv \\
& \ll T^{-2}\int_{-T}^{T}\frac{1}{1+|v|}\left(\int_0^{2T}|\psi(\sigma^*+i(u+v))|^2du\int_0^{2T}|\psi(\sigma^*-i(u-v))|^2du\right)^{\frac{1}{2}}dv \\
& \ll T^{-1+\varepsilon}\int_{-T}^{T}\frac{1}{1+|v|}dv \ll T^{-1+2\varepsilon}.
\end{aligned}$$

Now substituting (7.25) and (7.26) into (7.24) we get with $\varepsilon_1 = \frac{3\varepsilon}{2\alpha}$

$$(7.27)\qquad \int_x^{(1+\delta)x}|I(\lambda,P,Q,y)|^2\,dy \ll x^{1-\frac{1}{\alpha}+\varepsilon_1}P^{\frac{\mu}{\alpha}+\varepsilon_1}.$$

Applying a simple splitting argument to the interval $[M,N]$ with $M_0 = M, M_r = N, M_{j+1} \le (1+\delta)M_j$ $(j=0,\ldots,r-1; r \ll \log x)$ and Cauchy's inequality, it follows that

from (7.27)

$$\begin{aligned}
(7.28)\qquad \int_x^{(1+\delta)x} |I(\lambda, M, N, y)|^2\, dy &\ll \int_x^{(1+\delta)x} \left(\sum_{j=0}^{r-1} |I(\lambda, M_j, M_{j+1}, y)| \right)^2 dy \\
&= \sum_{j=0}^{r-1}\sum_{l=0}^{r-1} \int_x^{(1+\delta)x} |I(\lambda, M_j, M_{j+1}, y)|\, |I(\lambda, M_l, M_{l+1}, y)|\, dy \\
&\ll \sum_{j=0}^{r-1}\sum_{l=0}^{r-1} \sqrt{\int_x^{(1+\delta)x} |I(\lambda, M_j, M_{j+1}, y)|^2\, dy \times \int_x^{(1+\delta)x} |I(\lambda, M_l, M_{l+1}, y)|^2\, dy} \\
&= \left(\sum_{j=0}^{r-1} \sqrt{\int_x^{(1+\delta)x} |I(\lambda, M_j, M_{j+1}, y)|^2\, dy} \right)^2 \ll x^{1-\frac{1}{\alpha}+\varepsilon_1} \left(\sum_{j=0}^{r-1} M_j^{\frac{\mu}{2\alpha}+\frac{\varepsilon_1}{2}} \right)^2 .
\end{aligned}$$

Recall $\mu = 2\alpha(\lambda + \sigma^* + 1) - 1$. Now Lemma 9 follows from (7.28) at once. □

Lemma 10. *Let* $2(\lambda+\sigma^*) \neq -1$, $2(\lambda+\sigma^*+1) < \frac{1}{\alpha}$, $M \geq 1$, *and* $\delta > 0$ *with* $(1+\delta)^{\frac{1}{\alpha}}-1 < \frac{1}{4}$. *Then we have*

$$(7.29)\qquad \int_x^{(1+\delta)x} |I(\lambda, M, \infty, y)|^2\, dy \ll x^{1-\frac{1}{\alpha}+\varepsilon'} M^{2(\lambda+\sigma^*+1)-\frac{1}{\alpha}}.$$

Proof. We take $M_j = M(1 + \delta_1)^j$, $(j = 0, 1, \cdots, r - 1)$. The lemma is immediately obtained from (7.28) and let $r \to \infty$. □

Remark 11. In Tong's original proof (similar to Lemma 9) he used the second mean value theorem for complex integral, but we don't. Moreover, in both Lemma 9 and Lemma 10 the condition $2(\lambda + \sigma^*) \neq -1$ is not essential. In fact, we can use the second mean value theorem to remove this restriction.

8 The proof of Theorem 1

In this section we shall prove Theorem 1.

Suppose that $\mathcal{L}(s) \in \mathcal{S}^{\theta}_{real}$ is a function of degree $d \geq 2$ such that $0 \leq \theta \leq 1/2 - 1/2d$. So $\mathcal{L}(s)$ satisfies the functional equation (1.1) in Section 1. We assume $\sum_{j=1}^{L} \beta_j$ is real where β_j are the constants in (1.2). From (2.3) and (3.3)–(3.6), we have obviously $d = 2\alpha, \mu = \mu', \nu = \nu', \lambda = \lambda', \theta_\varrho = 1/2 - 1/2d + \varrho(1 - 1/d)$. We also have $\varphi(s) = \mathcal{L}(s)$ and $\psi(s) = \omega Q^{2s-1}\mathcal{L}(s)$, hence $b(n) = \omega Q^{-1}a(n)$, $\lambda_n = n$ and $\mu_n = Q^{-2}n$. Suppose that $1/2 \leq \sigma^* < 1$ satisfies (1.4) and (1.5).

Suppose T is a large parameter and $\delta > 0$ is a small positive constant. We shall evaluate the integral $\int_T^{(1+\delta)T} E^2(y)dy$.

8.1 Tong's formula of $E(y)$

We first show that

$$\int_1^T |E(y)|dy \ll T^{1+\frac{d^2-1}{2d^2}+\varepsilon}. \tag{8.1}$$

Since $\mathcal{L}(s)$ is a function in $\mathcal{S}_{real}^{\theta}$ of degree d, we have trivially that $\mathcal{L}(it) \ll (1+|t|)^{d/2+\varepsilon/2}$ and thus

$$\int_0^T |\mathcal{L}(it)|^2 dt \ll T^{d+1+\varepsilon}. \tag{8.2}$$

From (1.4), (8.2) and Lemma 8.3 of Ivić [19] we get for $0 \le \sigma \le \sigma^*$

$$\int_0^T |\mathcal{L}(\sigma+it)|^2 dt \ll T^{\frac{(d+1)(\sigma^*-\sigma)}{\sigma^*}+\frac{\sigma}{\sigma^*}+\varepsilon}. \tag{8.3}$$

Define $\hat{E}(y) = A(y) - \operatorname{Res}_{s=1} \mathcal{L}(s)y^s s^{-1}$. Then

$$E(y) = \hat{E}(y) - \mathcal{L}(0).$$

By Perron's formula we have for some $\sigma^* < \sigma < 1$ that

$$\hat{E}(y) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{L}(s)y^s s^{-1} ds.$$

From (8.3) and integration by parts we see that

$$\int_{-\infty}^{\infty} |\mathcal{L}(\sigma_0+it)|^2 |\sigma_0+it|^{-2} dt \ll 1$$

for $\sigma_0 = \frac{(d-1)\sigma^*+2\sigma^*\varepsilon}{d}$. So we have

$$\hat{E}(y) = \frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \mathcal{L}(s)y^s s^{-1} ds.$$

Replacing y in the above formula by $1/y$, and then using Parseval's identity (see the formula (A.5) of Ivić [19]) we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathcal{L}(\sigma_0+it)|^2 |\sigma_0+it|^{-2} dt = \int_0^{\infty} |\hat{E}(1/y)|^2 y^{2\sigma_0-1} dy = \int_0^{\infty} |\hat{E}(y)|^2 y^{-2\sigma_0-1} dy.$$

Thus we get for any $D > 1$ that

$$\int_D^{2D} |\hat{E}(y)|^2 dy \ll D^{2\sigma_0+1},$$

which with Cauchy's inequality gives

$$\int_D^{2D} |\hat{E}(y)|dy \ll D^{\sigma_0+1}.$$

By the splitting argument we get

$$\int_0^T |\hat{E}(y)|dy \ll T^{\sigma_0+1}.$$

Recalling $E(y) = \hat{E}(y) - \mathcal{L}(0)$ and (1.5) we get

$$\int_0^T |E(y)|dy \ll \int_0^T |\hat{E}(y)|dy + T \ll T^{\sigma_0+1} \ll T^{1+\frac{d^2-1}{2d^2}+\varepsilon},$$

namely (8.1) holds.

Now we use Theorem 7. Take $\varrho = 0$. From the first estimate of (1.3) we take $\omega_0 = \theta + \varepsilon$. From (8.1) we take $\omega_1 = (d^2-1)/2d^2 < 1/2$. Suppose $k > 1$ is a large but fixed integer which satisfies (6.2). Take $x = T, N = [T^{2d-1-\varepsilon}]$. M is a real number such that $T^\varepsilon \ll M \ll \sqrt{N}$ and $\hat{E}(M) = 0$ if $b(n) \geq 0 \ \ (n \geq 1)$ and $\hat{E}(M) \ll M^{\theta+\varepsilon}$ otherwise.

The truncated Tong-type formula of $E(y)$ was given in Theorem 7. Since we are not interested in the exact value of constants like κ_0, we may assume $Q = 1$ for simplicity of notation. Hence we may take $\mu_n = n$ in the rest of this section. Then from (6.5)–(6.11) of Theorem 7 we have

$$E(y) = \sum_{j=1}^{7} R_j(y), \tag{8.4}$$

where

$$R_1(y) = \kappa_0 y^{\frac{d-1}{2d}} \sum_{n \leq M} \frac{a(n)}{n^{\frac{d+1}{2d}}} \cos\left(h(yn)^{1/d} + c_0\pi\right),$$

$$R_2(y) = y^{\frac{d+1}{2d}} \mathrm{Re}\left(c_{00} I\left(-\frac{3}{2} + \frac{1}{2d}, M, N, y\right)\right),$$

$$R_3(y) = \sum_{\substack{0 \leq l,m \leq J \\ l+m>0}} \mathrm{Re}\left(c_{lm} I\left(-\frac{3}{2} + \frac{1}{2d} + \frac{l-m}{d}, M, N, y\right)\right) T^{-l} y^{-l+\frac{1}{2}+\frac{1}{2d}+\frac{l-m}{d}},$$

$$R_4(y) = \sum_{0 \leq l,m \leq k} \mathrm{Re}\left(c'_{lm} I\left(-\frac{3}{2} + \frac{1}{2d} - \frac{k+m}{d}, N, \infty, y + \frac{l}{T}\right)\right) T^k \left(y + \frac{l}{T}\right)^{k+\frac{1}{2}+\frac{1}{2d}-\frac{k+m}{d}},$$

$$\begin{aligned} R_5(y) \ll\; & T^{\frac{1}{2}-\frac{3}{2d}} M^{\max(\frac{1}{2}-\frac{3}{2d},0)} \log^A T + T^{-\frac{3}{2}+\frac{1}{2d}} M^{\frac{1}{2}+\frac{1}{2d}} \\ & + T^{-\frac{1}{2d}} M^{\omega_1-1-\frac{1}{2d}} + T^{(2d-1)(1+\omega_1)-2k+1+4k/d-3d} \end{aligned}$$

$$\ll T^{\frac{1}{2}-\frac{3}{2d}} M^{\max(\frac{1}{2}-\frac{3}{2d},0)} \log^A T,$$

$$R_6(y) \begin{cases} = 0 & \text{if } b(n) \geq 0 \\ \ll T^{\frac{1}{2}-\frac{1}{2d}} M^{\theta-\frac{1}{2}-\frac{1}{2d}+\varepsilon} & \text{otherwise,} \end{cases}$$

$$R_7(y) = E(y) - \int_{\mathbf{E}_k} E(\tilde{y}) d\Upsilon_k.$$

For simplicity, we write

$$E(y) = K_1 + K_2,$$

where

$$K_1 = R_1(y) + R_2(y),$$
$$K_2 = R_3(y) + R_4(y) + R_5(y) + R_6(y) + R_7(y).$$

8.2 Evaluation of $\int_T^{(1+\delta)T} K_1^2 dy$

In this subsection we shall evaluate $\int_T^{(1+\delta)T} K_1^2 dy$.

We write

(8.5)

$$\begin{aligned} R_1^2(y) &= \kappa_0^2 y^{\frac{d-1}{d}} \sum_{n,m \leq M} \frac{a(n)a(m)}{(nm)^{\frac{d+1}{2d}}} \cos\left(h(yn)^{1/d} + c_0\pi\right) \cos\left(h(ym)^{1/d} + c_0\pi\right) \\ &= \frac{\kappa_0^2}{2} y^{\frac{d-1}{d}} \sum_{n,m \leq M} \frac{a(n)a(m)}{(nm)^{\frac{d+1}{2d}}} \left(\cos\left(hy^{1/d}(n^{1/d} - m^{1/d})\right) + \cos\left(hy^{1/d}(n^{1/d} + m^{1/d}) + 2c_0\pi\right)\right) \\ &= W_1(y) + W_2(y) + W_3(y), \end{aligned}$$

where

$$W_1(y) = \frac{\kappa_0^2}{2} y^{\frac{d-1}{d}} \sum_{n \leq M} \frac{a^2(n)}{n^{\frac{d+1}{d}}},$$

$$W_2(y) = \frac{\kappa_0^2}{2} y^{\frac{d-1}{d}} \sum_{\substack{n,m \leq M \\ n \neq m}} \frac{a(n)a(m)}{(nm)^{\frac{d+1}{2d}}} \cos\left(hy^{1/d}(n^{1/d} - m^{1/d})\right),$$

$$W_3(y) = \frac{\kappa_0^2}{2} y^{\frac{d-1}{d}} \sum_{n,m \leq M} \frac{a(n)a(m)}{(nm)^{\frac{d+1}{2d}}} \cos\left(hy^{1/d}(n^{1/d} + m^{1/d}) + 2c_0\pi\right).$$

For $W_1(y)$, we have

$$\int_T^{(1+\delta)T} W_1(y) dy = \frac{\kappa_0^2}{2} \sum_{n \leq M} \frac{a^2(n)}{n^{\frac{d+1}{d}}} \int_T^{(1+\delta)T} y^{\frac{d-1}{d}} dy \tag{8.6}$$

$$= \frac{\kappa_0^2}{2} \sum_{n=1}^{\infty} \frac{a^2(n)}{n^{\frac{d+1}{d}}} \int_T^{(1+\delta)T} y^{\frac{d-1}{d}} dy + O(T^{2-1/d+\varepsilon} M^{-1/d}),$$

where in the last step we used the second estimate of (1.3).

By the first derivative test we have

$$\int_T^{(1+\delta)T} W_2(y) dy \ll T^{2-2/d} \sum_{\substack{n,m \le M \\ n \ne m}} \frac{|a(n)a(m)|}{(nm)^{\frac{d+1}{2d}} |n^{1/d} - m^{1/d}|}$$
$$= T^{2-2/d} (\Sigma_1 + \Sigma_2),$$

say, where

$$\Sigma_1 = \sum_{\substack{n,m \le M \\ |n^{1/d} - m^{1/d}| \ge (mn)^{1/2d}/10d}} \frac{|a(n)a(m)|}{(nm)^{\frac{d+1}{2d}} |n^{1/d} - m^{1/d}|},$$
$$\Sigma_2 = \sum_{\substack{n,m \le M \\ 0 < |n^{1/d} - m^{1/d}| < (mn)^{1/2d}/10d}} \frac{|a(n)a(m)|}{(nm)^{\frac{d+1}{2d}} |n^{1/d} - m^{1/d}|}.$$

For Σ_1 we have

$$\Sigma_1 \ll \sum_{\substack{n,m \le M \\ |n^{1/d} - m^{1/d}| \ge (mn)^{1/2d}/10d}} \frac{|a(n)a(m)|}{(nm)^{\frac{d+2}{2d}}}$$
$$\ll \left(\sum_{n \le M} \frac{|a(n)|}{n^{\frac{d+2}{2d}}} \right)^2 \ll M^{1-2/d+\varepsilon},$$

where we used the estimate $\sum_{n \le y} |a(n)| \ll y^{1+\varepsilon/2}$, which follows from the second estimate of (1.3) and Cauchy's inequality.

Next we consider the sum Σ_2. By Lagrange's mean value theorem we have $|n^{1/d} - m^{1/d}| \gg n_0^{1/d-1} |n-m|$ for some n_0 between n and m. If m, n are contained in the sum Σ_2, then $m \asymp n$ and hence

$$|n^{1/d} - m^{1/d}| \gg (mn)^{1/2d-1/2} |m-n|.$$

Thus

$$\Sigma_2 \ll \sum_{n \ne m} \frac{|a(n)a(m)|}{(nm)^{1/d} |n-m|}$$
$$\ll \sum_{n \ne m} \frac{1}{|n-m|} \left\{ \left(\frac{a(n)}{n^{1/d}} \right)^2 + \left(\frac{a(m)}{m^{1/d}} \right)^2 \right\}$$

$$
\begin{aligned}
&\ll \sum_{n\neq m} \frac{1}{|n-m|}\frac{a^2(n)}{n^{2/d}}\\
&\ll \sum_{n\leq M} \frac{a^2(n)}{n^{2/d}} \sum_{m\neq n} \frac{1}{|n-m|}\\
&\ll M^{1-2/d+\varepsilon}.
\end{aligned}
$$

From the above two estimates we get

$$
\int_T^{(1+\delta)T} W_2(y)dy \ll T^{2-2/d+\varepsilon}M^{1-2/d}. \tag{8.7}
$$

We have similarly that

$$
\int_T^{(1+\delta)T} W_3(y)dy \ll T^{2-2/d+\varepsilon}M^{1-2/d}. \tag{8.8}
$$

From (8.5), (8.6), (8.7) and (8.8) we have

$$
\begin{aligned}
\int_T^{(1+\delta)T} R_1^2(y)dy =&\frac{\kappa_0^2}{2}\sum_{n=1}^{\infty}\frac{a^2(n)}{n^{\frac{d+1}{d}}}\int_T^{(1+\delta)T} y^{\frac{d-1}{d}}dy\\
&+O(T^{2-1/d+\varepsilon}M^{-1/d}+T^{2-2/d+\varepsilon}M^{1-2/d}).
\end{aligned} \tag{8.9}
$$

From (1.5) we have $2\sigma^*-1-1/d<0$, so by Lemma 9 we have

$$
\begin{aligned}
\int_T^{(1+\delta)T} R_2^2(y)dy &\ll T^{\frac{d+1}{d}}\int_T^{(1+\delta)T}\left|I\left(-\frac{3}{2}+\frac{1}{2d},M,N,y\right)\right|^2 dy\\
&\ll T^{2-1/d+\varepsilon}\max_{M\leq P\leq N} P^{2\sigma^*-1-1/d}\\
&\ll T^{2-1/d+\varepsilon}M^{2\sigma^*-1-1/d}.
\end{aligned} \tag{8.10}
$$

By the definitions of $R_1(y)$ and $R_2(y)$ we have

$$
\begin{aligned}
&\int_T^{(1+\delta)T} R_1(y)R_2(y)dy\\
&=\operatorname{Re}\kappa_0c_{00}\int_T^{(1+\delta)T} yI\left(-\frac{3}{2}+\frac{1}{2d},M,N,y\right)\sum_{n\leq M}\frac{a(n)}{n^{\frac{d+1}{2d}}}\cos(h(yn)^{1/d}+c_0\pi)dy\\
&=\operatorname{Re}\kappa_0c_{00}(I_1+I_2),
\end{aligned}
$$

where

$$
I_1=\int_T^{(1+\delta)T} yI\left(-\frac{3}{2}+\frac{1}{2d},M,N,y\right)\sum_{n\leq M/2}\frac{a(n)}{n^{\frac{d+1}{2d}}}\cos(h(yn)^{1/d}+c_0\pi)dy,
$$

$$I_2 = \int_T^{(1+\delta)T} yI\left(-\frac{3}{2}+\frac{1}{2d}, M, N, y\right) \sum_{M/2<n\le M} \frac{a(n)}{n^{\frac{d+1}{2d}}} \cos(h(yn)^{1/d}+c_0\pi)dy.$$

By Lemma 8 we get

$$\begin{aligned}
I_1 &= \sum_{n\le M/2} \frac{a(n)}{n^{\frac{d+1}{2d}}} \int_T^{(1+\delta)T} yI\left(-\frac{3}{2}+\frac{1}{2d}, M, N, y\right) \cos(h(yn)^{1/d}+c_0\pi)dy \\
&\ll \sum_{n\le M/2} \frac{|a(n)|}{n^{\frac{d+1}{2d}}} \left| \int_T^{(1+\delta)T} yI\left(-\frac{3}{2}+\frac{1}{2d}, M, N, y\right) \cos(h(yn)^{1/d}+c_0\pi)dy \right| \\
&\ll \sum_{n\le M/2} \frac{|a(n)|}{n^{\frac{d+1}{2d}}} T^{2-3/2d+\varepsilon} \max_{M\le P\le N} P^{\sigma^*-1/2-1/d} \\
&\ll \sum_{n\le M/2} \frac{|a(n)|}{n^{\frac{d+1}{2d}}} T^{2-3/2d+\varepsilon} M^{\sigma^*-1/2-1/d} \\
&\ll T^{2-3/2d+\varepsilon} M^{\sigma^*-3/2d}.
\end{aligned}$$

By Cauchy's inequality

$$I_2 \ll T(V_1V_2)^{1/2},$$

where

$$\begin{aligned}
V_1 &= \int_T^{(1+\delta)T} \left| I\left(-\frac{3}{2}+\frac{1}{2d}, M, N, y\right) \right|^2 dy, \\
V_2 &= \int_T^{(1+\delta)T} \left(\sum_{M/2<n\le M} \frac{a(n)}{n^{\frac{d+1}{2d}}} \cos(h(yn)^{1/d}+c_0\pi) \right)^2 dy.
\end{aligned}$$

By Lemma 9 we get

$$V_1 \ll T^{1-2/d+\varepsilon} M^{2\sigma^*-1-1/d}.$$

By using the same approach of $\int_T^{(1+\delta)T} R_1^2(y)dy$, we get

$$V_2 \ll T^{1+\varepsilon} M^{-1/d} + T^{1-1/d+\varepsilon} M^{1-2/d}.$$

Thus

$$I_2 \ll T^{2-1/d+\varepsilon} M^{\sigma^*-1/2-1/d} + T^{2-3/2d+\varepsilon} M^{\sigma^*-3/2d}.$$

From the estimates of I_1 and I_2 we get

$$\int_T^{(1+\delta)T} R_1(y)R_2(y)dy \ll T^{2-1/d+\varepsilon} M^{\sigma^*-1/2-1/d} + T^{2-3/2d+\varepsilon} M^{\sigma^*-3/2d}. \tag{8.11}$$

Combining (8.9), (8.10) and (8.11) we get

$$\int_T^{(1+\delta)T} K_1^2 dy = \frac{\kappa_0^2}{2} \sum_{n=1}^{\infty} \frac{a^2(n)}{n^{\frac{d+1}{d}}} \int_T^{(1+\delta)T} y^{\frac{d-1}{d}} dy \tag{8.12}$$

$$+O(T^{2-1/d+\varepsilon}M^{2\sigma^*-1-1/d}+T^{2-2/d+\varepsilon}M^{1-2/d}),$$

where we used the following estimates (since $\sigma^*\geq 1/2$)

$$\begin{aligned}
&T^{2-1/d+\varepsilon}M^{-1/d}\ll T^{2-1/d+\varepsilon}M^{2\sigma^*-1-1/d},\\
&T^{2-1/d+\varepsilon}M^{\sigma^*-1/2-1/d}\ll T^{2-1/d+\varepsilon}M^{2\sigma^*-1-1/d},\\
&T^{2-3/2d+\varepsilon}M^{\sigma^*-3/2d}\ll \left(T^{2-1/d+\varepsilon}M^{2\sigma^*-1-1/d}\right)^{1/2}\left(T^{2-2/d+\varepsilon}M^{1-2/d}\right)^{1/2}.
\end{aligned}$$

8.3 Upper bound of $\int_T^{(1+\delta)T}K_2^2dy$

By Cauchy's inequality and Lemma 9, we have

$$\begin{aligned}
&\int_T^{(1+\delta)T}R_3^2(y)dy\\
&\ll \sum_{\substack{0\leq l,m\leq J\\ l+m>0}}T^{-2l}\int_T^{(1+\delta)T}y^{-2l+1+\frac{1}{d}+\frac{2l-2m}{d}}\left|I\left(-\frac{3}{2}+\frac{1}{2d}+\frac{l-m}{d},M,N,y\right)\right|^2dy\\
&\ll \sum_{\substack{0\leq l,m\leq J\\ l+m>0}}T^{-4l+1+\frac{1}{d}+\frac{2l-2m}{d}}\int_T^{(1+\delta)T}\left|I\left(-\frac{3}{2}+\frac{1}{2d}+\frac{l-m}{d},M,N,y\right)\right|^2dy\\
&\ll \sum_{\substack{0\leq l,m\leq J\\ l+m>0}}T^{-4l+1+\frac{1}{d}+\frac{2l-2m}{d}+1-\frac{2}{d}+\varepsilon}\max_{M\leq P\leq N}P^{2\sigma^*-1-\frac{1}{d}+\frac{2l-2m}{d}}\\
&=\Sigma_3+\Sigma_4,
\end{aligned}$$

say, where

$$\begin{aligned}
\Sigma_3&=\sum_{\substack{0\leq l\leq m\leq J\\ l+m>0}}T^{-4l+1+\frac{1}{d}+\frac{2l-2m}{d}+1-\frac{2}{d}+\varepsilon}\max_{M\leq P\leq N}P^{2\sigma^*-1-\frac{1}{d}+\frac{2l-2m}{d}},\\
\Sigma_4&=\sum_{0\leq m<l\leq J}T^{-4l+1+\frac{1}{d}+\frac{2l-2m}{d}+1-\frac{2}{d}+\varepsilon}\max_{M\leq P\leq N}P^{2\sigma^*-1-\frac{1}{d}+\frac{2l-2m}{d}}.
\end{aligned}$$

For Σ_3 we have by (1.5) that

$$\begin{aligned}
\Sigma_3&\ll \sum_{\substack{0\leq l\leq m\leq J\\ l+m>0}}T^{-4l+1+\frac{1}{d}+\frac{2l-2m}{d}+1-\frac{2}{d}+\varepsilon}M^{2\sigma^*-1-\frac{1}{d}+\frac{2l-2m}{d}}\\
&\ll T^{2-1/d+\varepsilon}M^{2\sigma^*-1-1/d}\sum_{1\leq m\leq J}T^{-2m/d}M^{-2m/d}\\
&\quad+T^{2-1/d+\varepsilon}M^{2\sigma^*-1-1/d}\sum_{1\leq l\leq J}T^{-4l+2l/d}M^{2l/d}\sum_{l\leq m\leq J}T^{-2m/d}M^{-2m/d}
\end{aligned}$$

$$\ll T^{2-3/d+\varepsilon}M^{2\sigma^*-1-3/d}.$$

For Σ_4 we have (since $\sigma^*\geq 1/2$ and recall $N=[T^{2d-1-\varepsilon}]$)

$$\begin{aligned}\Sigma_4 &\ll \sum_{0\leq m<l\leq J} T^{-4l+1+\frac{1}{d}+\frac{2l-2m}{d}+1-\frac{2}{d}+\varepsilon}N^{2\sigma^*-1-\frac{1}{d}+\frac{2l-2m}{d}}\\ &\ll T^{2-1/d+\varepsilon}N^{2\sigma^*-1-1/d}\sum_{0\leq m\leq J}T^{-2m/d}N^{-2m/d}\sum_{m+1\leq l\leq J}T^{-4l+\frac{2l}{d}}N^{\frac{2l}{d}}\\ &\ll T^{2-1/d+\varepsilon}N^{2\sigma^*-1-1/d}.\end{aligned}$$

From the above estimates we get

$$\int_T^{(1+\delta)T}R_3^2(y)dy\ll T^{2-3/d+\varepsilon}M^{2\sigma^*-1-3/d}+T^{2-1/d+\varepsilon}N^{2\sigma^*-1-1/d}. \tag{8.13}$$

By Cauchy's inequality and Lemma 10, we get (recall that $N=[T^{2d-1-\varepsilon}]$)

(8.14)

$$\begin{aligned}&\int_T^{(1+\delta)T}R_4^2(y)dy\\ &\ll \sum_{0\leq l,m\leq k}T^{2k}\int_T^{(1+\delta)T}(y+\frac{l}{T})^{2k+1+\frac{1}{d}-\frac{2k+2m}{d}}\left|I\left(-\frac{3}{2}+\frac{1}{2d}-\frac{k+m}{d},N,\infty,y+\frac{l}{T}\right)\right|^2dy\\ &\ll \sum_{0\leq l,m\leq k}T^{4k+1+\frac{1}{d}-\frac{2k+2m}{d}+1-\frac{2}{d}+\varepsilon}N^{2\sigma^*-1-\frac{1}{d}-\frac{2k+2m}{d}}\\ &\ll T^{4k+2-\frac{1}{d}-\frac{2k}{d}+\varepsilon}N^{2\sigma^*-1-\frac{1}{d}-\frac{2k}{d}}\ll T^{2-\frac{1}{d}+\varepsilon}N^{2\sigma^*-1-\frac{1}{d}}.\end{aligned}$$

For $R_5(y)$ we have trivially that

$$\int_T^{(1+\delta)T}R_5^2(y)dy\ll\begin{cases}T^{1/2}, & \text{if } d=2,\\ T^{2-3/d+\varepsilon}M^{1-3/d}, & \text{if } d\geq 3.\end{cases} \tag{8.15}$$

For $R_6(y)$ we have trivially that

$$\int_T^{(1+\delta)T}R_6^2(y)dy\ll T^{2-1/d+\varepsilon}M^{2\theta-1-1/d+\varepsilon}. \tag{8.16}$$

Next we estimate $\int_T^{(1+\delta)T}R_7^2(y)dy$. Trivially we have

$$\begin{aligned}\int_T^{(1+\delta)T}R_7^2(y)dy&\ll\sum_{T\leq n\leq(1+\delta)T+1}\int_{n-1}^{n}R_7^2(y)dy\\ &\ll\sum_{T\leq n\leq(1+\delta)T+1}\int_{n-1}^{n-k/T}R_7^2(y)dy+\sum_{T\leq n\leq(1+\delta)T+1}\int_{n-k/T}^{n}R_7^2(y)dy.\end{aligned}$$

It is easy to see that

$$R_7(y) = E(y) - \int_{\mathbf{E}_k} E(\tilde{y}) d\mathrm{Y}_k = \int_{\mathbf{E}_k} (E(y) - E(\tilde{y})) d\mathrm{Y}_k.$$

Suppose $n-1 < y \leq n$. If $n-1 < y \leq \tilde{y} < n$, then by the definition of $E(u)$ we get

$$E(\tilde{y}) - E(y) = Q(y) - Q(\tilde{y}) \ll T^{-1} \max_{y \ll u \ll \tilde{y}} Q'(u) \ll T^{-1} \log^{m_{\mathcal{L}}} T;$$

otherwise if $\tilde{y} \geq n$, then

$$E(\tilde{y}) - E(y) = Q(y) - Q(\tilde{y}) + a(n)/2 \ll T^{-1} \log^{m_{\mathcal{L}}} T + |a(n)|.$$

Thus we have

$$\sum_{T \leq n \leq (1+\delta)T+1} \int_{n-1}^{n-k/T} R_7^2(y) dy \ll T^{-1} \log^{2m_{\mathcal{L}}} T,$$
$$\sum_{T \leq n \leq (1+\delta)T+1} \int_{n-k/T}^{n} R_7^2(y) dy \ll T^{-1} \log^{2m_{\mathcal{L}}} T + T^{-1} \sum_{n \ll T} a^2(n) \ll T^{\varepsilon}.$$

Hence we get

$$\int_T^{(1+\delta)T} R_7^2(y) dy \ll T^{\varepsilon}. \tag{8.17}$$

The first term in the right hand side of (8.13) is absorbed in the right hand side of (8.15). Furthermore from the assumption $M \ll N^{1/2}$ and $1/2 \leq \sigma^* < 1/2 + 1/2d$, we have

$$N^{2\sigma^* - 1 - 1/d} \ll M^{2(2\sigma^* - 1 - 1/d)}$$

and also

$$M^{2\theta - 1 - 1/d} \ll M^{2(2\sigma^* - 1 - 1/d)}$$

by $\theta < 1/2 - 1/2d$. Hence from (8.13)–(8.17) we have

$$\int_T^{(1+\delta)T} K_2^2 dy \ll T^{2-1/d+\varepsilon} M^{2(2\sigma^* - 1 - 1/d)} + \begin{cases} T^{1/2}, & \text{if } d = 2, \\ T^{2-3/d+\varepsilon} M^{1-3/d}, & \text{if } d \geq 3. \end{cases} \tag{8.18}$$

8.4 Proof of Theorem 1

Now we take M such that

$$M \asymp T^{\frac{1}{2d(1-\sigma^*)-1}}, \tag{8.19}$$

and

$$\hat{E}(M) \begin{cases} = 0, & \text{if } b(n) \geq 0 \ (n \geq 1), \\ \ll T^{\theta+\varepsilon}, & \text{otherwise.} \end{cases}$$

By noting that $\sigma^* = 1/2$ for $d = 2$, we can see easily that $M \ll N^{1/2}$ for any $d \geq 2$. Then the formula (8.12) becomes

$$\int_T^{(1+\delta)T} K_1^2 dy = \frac{\kappa_0^2}{2} \sum_{n=1}^{\infty} \frac{a^2(n)}{n^{\frac{d+1}{d}}} \int_T^{(1+\delta)T} y^{\frac{d-1}{d}} dy + O(T^{2-\frac{3-4\sigma^*}{2d(1-\sigma^*)-1}+\varepsilon}). \tag{8.20}$$

From (8.18), (8.20) and Cauchy's inequality we get

$$\begin{aligned} \int_T^{(1+\delta)T} K_1 K_2 dy &\ll T^{2-1/d+\varepsilon} M^{(2\sigma^*-1-1/d)} + \begin{cases} T^{\varepsilon}, & \text{if } d = 2 \\ T^{2-2/d+\varepsilon} M^{(1-3/d)/2}, & \text{if } d \geq 3 \end{cases} \\ &\ll T^{2-\frac{3-4\sigma^*}{2d(1-\sigma^*)-1}+\varepsilon}, \end{aligned} \tag{8.21}$$

where the last inequality follows from the choice of M (8.19). It is also seen easily that

$$\int_T^{(1+\delta)T} K_2^2 dy \ll T^{2-\frac{3-4\sigma^*}{2d(1-\sigma^*)-1}+\varepsilon} \tag{8.22}$$

by (8.18) and the choice of M. Hence from (8.20), (8.21) and (8.22) we get

$$\int_T^{(1+\delta)T} E^2(y) dy = \frac{\kappa_0^2}{2} \sum_{n=1}^{\infty} \frac{a^2(n)}{n^{\frac{d+1}{d}}} \int_T^{(1+\delta)T} y^{\frac{d-1}{d}} dy + O(T^{2-\frac{3-4\sigma^*}{2d(1-\sigma^*)-1}+\varepsilon}). \tag{8.23}$$

Finally from (8.23) we get

$$\begin{aligned} \int_0^T E^2(y) dy &= \sum_{0 \leq j \leq \frac{\log T}{\delta}} \int_{\frac{T}{(1+\delta)^{j+1}}}^{\frac{T}{(1+\delta)^j}} E^2(y) dy + O(T^{2\delta}) \\ &= \sum_{0 \leq j \leq \frac{\log T}{\delta}} \frac{\kappa_0^2}{2} \sum_{n=1}^{\infty} \frac{a^2(n)}{n^{\frac{d+1}{d}}} \int_{\frac{T}{(1+\delta)^{j+1}}}^{\frac{T}{(1+\delta)^j}} y^{\frac{d-1}{d}} dy \\ &\quad + \sum_{0 \leq j \leq \frac{\log T}{\delta}} O\left(\left(\frac{T}{(1+\delta)^j} \right)^{2-\frac{3-4\sigma^*}{2d(1-\sigma^*)-1}+\varepsilon} \right) + O(T^{2\delta}) \\ &= \frac{\kappa_0^2}{2} \sum_{n=1}^{\infty} \frac{a^2(n)}{n^{\frac{d+1}{d}}} \int_0^T y^{\frac{d-1}{d}} dy + O\left(T^{2-\frac{3-4\sigma^*}{2d(1-\sigma^*)-1}+\varepsilon} \right). \end{aligned}$$

8.5 Proofs of Corollaries 1 and 2

We first prove Corollary 1. Suppose that $0 \leq \theta \leq 1/4$ is a real number and $\mathcal{L}(s) \in \mathcal{S}_{real}^{\theta}$ is a function of degree 2. In this case it is well-known that

$$\int_0^T |\mathcal{L}(1/2 + it)|^2 dt \ll T^{1+\varepsilon}, \tag{8.24}$$

namely, we have $\sigma^* = 1/2 < 3/4$. So Corollary 1 follows from Theorem 1.

Next we prove Corollary 2. Suppose that $0 \le \theta \le 1/3$ is a real number and $\mathcal{L}(s) \in \mathcal{S}^{\theta}_{real}$ is a function of degree 3 which can be written as $\mathcal{L}(s) = \mathcal{L}_1(s)\mathcal{L}_2(s)$, where $\mathcal{L}_1(s) \in \mathcal{S}^{\theta}_{real}$ is a function of degree 1 and $\mathcal{L}_2(s) \in \mathcal{S}^{\theta}_{real}$ is a function of degree 2. We can show that

$$\int_0^T |\mathcal{L}(5/8 + it)|^2 dt \ll T^{1+\varepsilon}. \tag{8.25}$$

Hence we can take $\sigma^* = 5/8$ in (1.4) and the assertion (1.8) follows from Theorem 1. Furthermore if we assume (1.9), we can show that

$$\int_0^T |\mathcal{L}(7/12 + it)|^2 dt \ll T^{1+\varepsilon}. \tag{8.26}$$

Hence we can take $\sigma^* = 7/12$ in this case and we get a better assertion (1.10).

In order to prove (8.25) or (8.26), we follow a method of Ivić [22]. In [22], Ivić treated the automorphic L-function attached to a cusp form over $SL_2(\mathbb{Z})$. Our proof for the general case is due to Ivić with some modifications. So here we give a detailed proof.

Lemma 11. *Let $t_1 < \cdots < t_R$ be real numbers such that $T \le t_r \le 2T$ for $r = 1, 2, \ldots, R$ and $|t_r - t_s| \ge \log^4 T$ for $1 \le s \ne r \le R$. Suppose V and $1 \ll M \ll T^C$ $(C > 0)$ are large parameters such that*

$$T^{\varepsilon} < V \le \left| \sum_{M < n \le 2M} a(n) n^{-\sigma - it_r} \right|$$

with

$$\sum_{M \le n \le 2M} |a(n)|^2 \ll M^{1+\varepsilon}, \tag{8.27}$$

then we have

$$R \ll (M^{2-2\sigma} V^{-2} + T V^{-\frac{2}{3-4\sigma}}) T^{\varepsilon}$$

for $1/2 < \sigma \le 2/3$.

Remark to Lemma 11. This is the case $1/2 < \sigma \le 2/3$ of Lemma 8.2 of Ivić [19], where Ivić supposed the condition $a(n) \ll M^{\varepsilon}$ for $M < n \le 2M$. However we see easily from Ivić's argument that this condition can be relaxed to the condition (8.27).

Lemma 12. *Let $0 \le \theta \le 1/3$ be a fixed real number and let $\mathfrak{L}(s)$ be an element of $\mathcal{S}^{\theta}_{real}$ with degree 2. Then we have*

$$\int_0^T |\mathfrak{L}(\sigma + it)|^{1/(1-\sigma)} dt \ll T^{1+\varepsilon} \tag{8.28}$$

for $1/2 < \sigma < 2/3$. *Especially we have*

$$\int_0^T |\mathfrak{L}(5/8+it)|^{8/3} dt \ll T^{1+\varepsilon}. \tag{8.29}$$

Furthermore if we assume

$$\int_0^T |\mathfrak{L}(1/2+it)|^{6} dt \ll T^{2+\varepsilon}, \tag{8.30}$$

we have

$$\int_0^T |\mathfrak{L}(\sigma+it)|^{2/(3-4\sigma)} dt \ll T^{1+\varepsilon} \tag{8.31}$$

for $1/2 < \sigma < 5/8$. *Especially we have*

$$\int_0^T |\mathfrak{L}(7/12+it)|^{3} dt \ll T^{1+\varepsilon}. \tag{8.32}$$

Proof. We follow the approach of Ivić [22]. Suppose $T < t_1 < t_2 < \cdots < t_R < 2T$ such that

$$|\mathfrak{L}(\sigma+it_r)| \geq V \gg T^{\varepsilon} \tag{8.33}$$

and

$$|t_r - t_s| \gg \log^4 T \ (1 \leq r \neq s \leq R).$$

Suppose that

$$\mathfrak{L}(s) = \sum_{n\geq 1} a(n) n^{-s}, \quad \mathrm{Re}\, s > 1.$$

From the formula (see (A7) of Ivić [19])

$$e^{-v} = (2\pi i)^{-1} \int_{b-i\infty}^{b+i\infty} \Gamma(w) v^{-w} dw \ (b, v > 0),$$

we have

$$\sum_{n\geq 1} a(n) e^{-n/Y} n^{-s} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(w) Y^w \mathfrak{L}(s+w) dw, \tag{8.34}$$

where $1 \ll Y \ll T^A$ is a parameter to be chosen later and $A > 0$ is an arbitrary constant. Let s be any one of $\sigma + it_r$ with $\sigma > 1/2$ in (8.34). Moving the line of integration to $1/2 - \sigma$ in (8.34) we get by the residue theorem that

$$\mathfrak{L}(s) = \sum_{n\geq 1} a(n) e^{-n/Y} n^{-s} - \frac{1}{2\pi i} \int_{1/2-\sigma-i\infty}^{1/2-\sigma+i\infty} \Gamma(w) Y^w \mathfrak{L}(s+w) dw + O(T^{-B}),$$

where $B > 1$ is a large fixed constant. So we get

$$\mathfrak{L}(\sigma + it_r) \ll \left| \sum_{n \le Y \log^2 T} a(n) e^{-n/Y} n^{-\sigma - it_r} \right| + Y^{1/2-\sigma} \int_{-\log^2 T}^{\log^2 T} |\mathfrak{L}(1/2 + it_r + iv)| dv + 1,$$

which implies that if (8.33) holds, we must have

$$V \ll \left| \sum_{n \le Y \log^2 T} a(n) e^{-n/Y} n^{-\sigma - it_r} \right| \ll \log T \max_{N_1 \le \frac{Y \log^2 T}{2}} \left| \sum_{N_1 < n \le 2N_1} a(n) e^{-n/Y} n^{-\sigma - it_r} \right| \tag{8.35}$$

or

$$V \ll Y^{1/2-\sigma} \int_{-\log^2 T}^{\log^2 T} |\mathfrak{L}(1/2 + it_r + iv)| dv. \tag{8.36}$$

Let $\mathcal{A}_1$ be the set of points t_r which satisfy (8.35) and $\mathcal{A}_2$ the set of points t_r which satisfy (8.36). We also let $R_1 = |\mathcal{A}_1|$ and $R_2 = |\mathcal{A}_2|$, whence $R = R_1 + R_2$.

For R_1 we have

$$R_1 \ll T^{\varepsilon} (Y^{2-2\sigma} V^{-2} + T V^{-2/(3-4\sigma)}) \tag{8.37}$$

by Lemma 11.

Now we are going to estimate R_2. Since $t_r \in \mathcal{A}_2$ satisfies (8.36) we have by Cauchy's inequality

$$V \ll Y^{1/2-\sigma} \log T \left(\int_{-\log^2 T}^{\log^2 T} |\mathfrak{L}(1/2 + it_r + iv)|^2 dv \right)^{1/2}.$$

Squaring both sides and summing up over $t_r \in \mathcal{A}_2$, we get

$$\begin{aligned} R_2 &\ll Y^{1-2\sigma} V^{-2} (\log T)^2 \sum_{t_r \in \mathcal{A}_2} \int_{-\log^2 T}^{\log^2 T} |\mathfrak{L}(1/2 + it_r + iv)|^2 dv \\ &\ll Y^{1-2\sigma} V^{-2} (\log T)^2 \int_{T - \log^2 T}^{2T + \log^2 T} |\mathfrak{L}(1/2 + iv)|^2 dv \\ &\ll Y^{1-2\sigma} V^{-2} T^{1+\varepsilon}, \end{aligned} \tag{8.38}$$

where in the last inequality we used the mean square estimate (8.24) for $\mathfrak{L}(1/2 + it)$ of degree 2. Hence as a first estimate we get, from (8.37) and (8.38),

$$R \ll T^{\varepsilon} \left(Y^{2-2\sigma} V^{-2} + T V^{-2/(3-4\sigma)} + Y^{1-2\sigma} V^{-2} T \right). \tag{8.39}$$

We shall derive another estimate for R_2. It is convenient to consider the conditional case first, so we assume (8.30). This time we use Hölder's inequality in (8.36). Thus we have

$$V \ll Y^{1/2-\sigma}(\log T)^{5/3}\left(\int_{-\log^2 T}^{\log^2 T}|\mathfrak{L}(1/2+it_r+iv)|^6 dv\right)^{1/6},$$

and

$$\begin{aligned}
R_2 &\ll Y^{3-6\sigma}V^{-6}(\log T)^{10}\sum_{t_r\in\mathcal{A}_2}\int_{-\log^2 T}^{\log^2 T}|\mathfrak{L}(1/2+it_r+iv)|^6 dv \\
&\ll Y^{3-6\sigma}V^{-6}(\log T)^{10}\int_{T-\log^2 T}^{2T+\log^2 T}|\mathfrak{L}(1/2+iv)|^6 dv \\
&\ll Y^{3-6\sigma}V^{-6}T^{2+\varepsilon}.
\end{aligned} \tag{8.40}$$

Hence (8.37) and (8.40) give the second estimate

$$R \ll T^{\varepsilon}\left(Y^{2-2\sigma}V^{-2}+TV^{-2/(3-4\sigma)}+Y^{3-6\sigma}V^{-6}T^2\right). \tag{8.41}$$

If $V \le T^{(3-4\sigma)/4}$, we apply (8.39) with $Y=T$. Then

$$\begin{aligned}
R &\ll (Y^{2-2\sigma}V^{-2}+TV^{-\frac{2}{3-4\sigma}}+TY^{1-2\sigma}V^{-2})T^{\varepsilon} \\
&\ll (T^{2-2\sigma}V^{-2}+TV^{-\frac{2}{3-4\sigma}})T^{\varepsilon} \\
&\ll T^{1+\varepsilon}V^{-\frac{2}{3-4\sigma}}
\end{aligned}$$

by recalling the condition $V \le T^{(3-4\sigma)/4}$. Next if $V \ge T^{(3-4\sigma)/4}$, we apply (8.41) with $Y = T^{2/(4\sigma-1)}V^{-4/(4\sigma-1)}+1$ and get

$$\begin{aligned}
R &\ll (Y^{2-2\sigma}V^{-2}+TV^{-\frac{2}{3-4\sigma}}+T^2Y^{3-6\sigma}V^{-6})T^{\varepsilon} \\
&\ll (TV^{-\frac{2}{3-4\sigma}}+T^{\frac{4-4\sigma}{4\sigma-1}}V^{-\frac{6}{4\sigma-1}})T^{\varepsilon} \\
&\ll T^{1+\varepsilon}V^{-\frac{2}{3-4\sigma}}
\end{aligned}$$

if $T^{\frac{5-8\sigma}{4\sigma-1}} \le V^{\frac{4(5-8\sigma)}{(4\sigma-1)(3-4\sigma)}}$, which is true when $V \ge T^{(3-4\sigma)/4}$ and $1/2<\sigma\le 5/8$. Summing up, if $1/2<\sigma\le 5/8$,

$$R \ll TV^{-2/(3-4\sigma)}$$

holds true for all V, completing the proof of (8.31).

Next we shall prove (8.28). We use the fourth-power mean value theorem:

$$\int_0^T |\mathfrak{L}(1/2+it)|^4 dt \ll T^{2+\varepsilon}, \tag{8.42}$$

which is known to be true unconditionally (see e.g. Kanemitsu, Sankaranarayanan and Tanigawa [29]). By the similar way we have

$$R \ll T^{\varepsilon} \left(Y^{2-2\sigma} V^{-2} + T V^{-2/(3-4\sigma)} + Y^{2-4\sigma} V^{-4} T^{2} \right) \tag{8.43}$$

from (8.42). If $V \le T^{1-\sigma}$, we take $Y = T$ in (8.39) and get

$$R \ll T^{\varepsilon} \left(T^{2-2\sigma} V^{-2} + T V^{-2/(3-4\sigma)} \right) \ll T^{1+\varepsilon} V^{-1/(1-\sigma)}.$$

On the other hand if $V \ge T^{1-\sigma}$, we take $V = (TV^{-1})^{1/\sigma}$ in (8.43) and get

$$R \ll T^{2-2\sigma)/\sigma} V^{-2/\sigma} + T V^{-2/(3-4\sigma)} \ll T V^{-1/(1-\sigma)}.$$

Therefore the estimate

$$R \ll T^{1+\varepsilon} V^{-1/(1-\sigma)}$$

holds true for all V for $1/2 < \sigma < 2/3$. This completes the proof of (8.28).

Now we prove (8.25) and (8.26). Recall that $\mathcal{L}(s) = \mathcal{L}_1(s)\mathcal{L}_2(s)$, where $\mathcal{L}_1(s) \in \mathcal{S}^{\theta}_{real}$ is a function of degree 1 and $\mathcal{L}_2(s) \in \mathcal{S}^{\theta}_{real}$ is a function of degree 2. By the work of Kaczorowski and Perelli [27], it is known that the functions of degree one in the Selberg class are the Riemann zeta-function $\zeta(s)$ and the shifts $L(s+i\vartheta, \chi)$ of Dirichlet L-functions attached to primitive characters χ with $\vartheta \in \mathbb{R}$. So we have

$$\int_0^T |\mathcal{L}_1(5/8+it)|^8 dt \ll T \log^4 T$$

and

$$\int_0^T |\mathcal{L}_2(5/8+it)|^{8/3} dt \ll T^{1+\varepsilon}.$$

Hence by Hölder's inequality and (8.29)

$$\begin{aligned}
\int_0^T |\mathcal{L}(5/8+it)|^2 dt &= \int_0^T |\mathcal{L}_1(5/8+it)\mathcal{L}_2(5/8+it)|^2 dt \\
&\ll \left(\int_0^T |\mathcal{L}_1(5/8+it)|^{2\cdot 4} dt \right)^{1/4} \left(\int_0^T |\mathcal{L}_2(5/8+it)|^{2\cdot 4/3} dt \right)^{3/4} \\
&\ll T^{1+\varepsilon}.
\end{aligned}$$

Assume that (1.9) holds. Then by

$$\int_0^T |\mathcal{L}_1(7/12+it)|^6 dt \ll T^{1+\varepsilon}$$

and

$$\int_0^T |\mathcal{L}_2(7/12+it)|^3 dt \ll T^{1+\varepsilon}$$

(this comes from (8.32)), we have

$$\int_0^T |\mathcal{L}(7/12+it)|^2 dt \ll T^{1+\varepsilon}$$

similarly. This completes the proof of Corollary 2.

9 Proofs of Theorem 2 and Theorem 3

We shall follow Heath-Brown's argument [15] to prove Theorem 2 and Theorem 3. We first quote some results from [15]. The following Hypothesis (H), Lemma 13 and Lemma 14 are Hypothesis (H), Theorem 5 and Theorem 6 of [15], respectively.

Hypothesis (H): Let $\mathcal{M}(t)$ be a real valued function, $a_1(t), a_2(t), \cdots$, be continuous real valued functions of period 1, and suppose there are non-zero constants $\gamma_1, \gamma_2, \cdots$ such that

$$\lim_{N\to\infty} \limsup_{T\to\infty} \frac{1}{T}\int_0^T \min\left(1, \left|\mathcal{M}(t) - \sum_{n\le N} a_n(\gamma_n t)\right|\right) dt = 0.$$

Lemma 13. *Suppose $\mathcal{M}(t)$ satisfies* (H) *and suppose that the constants γ_i are linearly independent over $\mathbb{Q}$. Suppose further*

$$\int_0^1 a_n(t)dt = 0 \quad (n \in \mathbb{N}),$$
$$\sum_{n=1}^{\infty}\int_0^1 a_n^2(t)dt < \infty,$$

and there is a constant $\mu > 1$ such that

$$\max_{t\in[0,1]} |a_n(t)| \ll n^{1-\mu},$$
$$\lim_{n\to\infty} n^{\mu}\int_0^1 a_n^2(t)dt = \infty.$$

Then $\mathcal{M}(t)$ has a distribution function $f(\alpha)$ with the properties described in Theorem 2.

Lemma 14. *Suppose $\mathcal{M}(t)$ satisfies* (H) *and that*

$$\int_0^T |\mathcal{M}(t)|^K dt \ll T$$

holds for some positive number K. Then for any real number $k \in [0, K)$, the limit

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T |\mathcal{M}(t)|^K dt$$

exists.

Suppose $T \le y \le (1+\delta)T$, define

$$\mathcal{M}(y) = \kappa_0^{-1}(2\pi/h)^{-\frac{d-1}{2}} y^{-\frac{d-1}{2}} E((2\pi)^d h^{-d} y^d),$$
$$a_n(y) = \frac{q_d(n)}{n^{\frac{d+1}{2d}}} \sum_{m=1}^{\infty} \frac{a(nm^d)}{m^{\frac{d+1}{2}}} \cos(2\pi y m + c_0\pi),$$

and

$$\gamma_n = n^{1/d},$$

where $q_d(n)$ is 1 if n is d-free and 0 otherwise.

It is easy to see that $a_n(y)$ satisfies all conditions of Lemma 13 for any fixed constant $1 + 1/d < \mu < 3/2 + 1/2d$.

Now we suppose $M_0 \asymp T^{\frac{d}{2d(1-\sigma^*)-1}}$ such that $E(M_0) \ll T^{d\theta+\varepsilon}$ and $N_0 = T^{d(2d-1)-\varepsilon}$. By (8.4) we have

$$\mathcal{M}(y) = \mathcal{M}_1(y) + \mathcal{M}_2(y),$$
$$\mathcal{M}_1(y) = \sum_{1 \le n \le M_0} \frac{a(n)}{n^{\frac{d+1}{2d}}} \cos\left(2\pi y n^{1/d} + c_0\pi\right),$$
$$\mathcal{M}_2(y) = \kappa_0^{-1}(2\pi/h)^{-\frac{d-1}{2}} y^{-\frac{d-1}{2}} \sum_{j=2}^{7} R_j((2\pi)^d h^{-d} y^d),$$

where $R_j(y)$ is defined in Subsection 8.1.

It is easy to see that for any integer $N_1 \le M_0$ we have

$$\begin{aligned}
&|\mathcal{M}(y) - \sum_{n \le N_1} a_n(\gamma_n y)| \\
&\ll \left| {\sum_{n \le M_0}}^{*} \frac{a(n)}{n^{\frac{d+1}{2d}}} \cos(2\pi y n^{1/d} + c_0\pi) \right| + \sum_{n \le N_1} \frac{1}{n^{\frac{d+1}{2d}}} \sum_{m > (M_0 n^{-1})^{1/d}} \frac{|a(nm^d)|}{m^{(d+1)/2}} + |\mathcal{M}_2(y)| \\
&\ll \left| {\sum_{n \le M_0}}^{*} \frac{a(n)}{n^{\frac{d+1}{2d}}} \cos(2\pi y n^{1/d} + c_0\pi) \right| + N_1^{1-1/d} M_0^{\theta + 1/2d - 1/2 + \varepsilon} + |\mathcal{M}_2(y)|,
\end{aligned}$$

where $\sum^*$ means that n has a d-free kernel great that N_1, and for the sum involving $a(nm^d)$ in the above formula we used the first estimate of (1.3) and the condition $\theta < 1/2 - 1/2d$.

Similarly to the approach of (8.9) we have

$$\int_T^{(1+\delta)T} \left| {\sum_{n \le M_0}}^{*} \frac{a(n)}{n^{\frac{d+1}{2d}}} \cos(2\pi y n^{1/d} + c_0\pi) \right|^2 dy$$

$$
\begin{aligned}
&\ll T \sum_{n\le M_0}{}^{*} \frac{a^2(n)}{n^{\frac{d+1}{d}}} + \sum_{\substack{n\neq m\\ n,m\le M_0}} \frac{|a(n)a(m)|}{(nm)^{\frac{d+1}{2d}}|n^{1/d}-m^{1/d}|}\\
&\ll T \sum_{n>N_1} \frac{a^2(n)}{n^{\frac{d+1}{d}}} + M_0^{1-2/d}\log^\eta M_0\\
&\ll TN_1^{-1/d+\varepsilon} + M_0^{1-2/d+\varepsilon}.
\end{aligned}
$$

Let $\mathcal{M}_2^*(y) = \sum_{j=2}^7 R_j(y)$, then $\mathcal{M}_2(y) = \mathcal{M}_2^*((2\pi)^d h^{-d} y^d)$. From (8.10), (8.13)–(8.17) we get

$$
\int_T^{(1+\delta)T} \mathcal{M}_2^{*2}(y)dy \ll \sum_{j=2}^7 \int_T^{(1+\delta)T} R_j^2(y)dy \ll T^{1-\frac{d+1-2d\sigma^*}{2d(1-\sigma^*)-1}+\varepsilon},
$$

which combining a change of variable implies that

$$
\int_T^{(1+\delta)T} \mathcal{M}_2^2(y)dy \ll T^{1/d-\frac{d+1-2d\sigma^*}{2d(1-\sigma^*)-1}+\varepsilon} \ll T^{-c}
$$

for some positive constant $c>0$.

From the above estimates and Cauchy's inequality we get

$$
\limsup_{T\to\infty} \frac{1}{T}\int_T^{2T} \Big|\mathcal{M}(y) - \sum_{n\le N_1} a_n(\gamma_n y)\Big|^2 dy \ll N_1^{-1/d+\varepsilon}
$$

and whence Hypothesis (H) follows. From Lemma 13 with $\mu = 5/4+3/4d$ we get Theorem 2.

From Theorem 1 and integration by parts we get easily that

$$
\int_0^T \left(y^{-\frac{d-1}{2d}}E(y)\right)^2 dy \ll T,
$$

hence Theorem 3 follows from Lemma 14 by taking $\mathcal{M}(y) = y^{-\frac{d-1}{2d}}E(y)$.

10 Some applications of our main results

In this section we consider some applications of our results.

10.1 Some functions of degree 2

Example 1: The Dirichlet divisor problem. The first example is $\zeta^2(s)\in\mathcal{S}^0_{real}$. Since

$$
\zeta^2(s) = \sum_{n=1}^\infty d(n)n^{-s} \qquad \operatorname{Re} s > 1,
$$

where $d(n)$ is the divisor function, this is the Dirichlet divisor problem. This problem is fundamental for all theory and hence has a long history. Let us review some of them. Dirichlet first proved that the error term

$$\Delta(x) := \sum_{n\le x}{}' d(n) - x\log x - (2\gamma-1)x, \quad x\ge 2$$

satisfies $\Delta(x) = O(x^{1/2})$. The latest result is due to Huxley [17], who showed that

$$\Delta(x) \ll x^{131/416}(\log x)^{26947/8320}.$$

Cramér [6] proved the classical mean-square result

$$\int_1^T \Delta^2(x)dx \;=\; \frac{(\zeta(3/2))^4}{6\pi^2\zeta(3)}T^{3/2} + F(T)$$

with $F(T) = O(T^{5/4+\varepsilon})$. The estimate $F(T) = O(T^{5/4+\varepsilon})$ was improved to $O(T\log^5 T)$ by Tong [49], from which the present work originates. Tong's result was improved to $O(T\log^4 T)$ by Preissmann [40], and recently to $O(T\log^3 T\log\log T)$ by Lau and Tsang [34]. For the asymptotic formulae of higher power moments of $\Delta(x)$ see, for example, the papers of Ivić and Sargos [24], Tsang [51] and Zhai [55]. For a survey of the Dirichlet divisor problem, see for example, Krätzel [31], Ivić [19] or Tsang [50].

Corollary 1 gives $F(T) = O(T^{1+\varepsilon})$. Since our setting of the problem is of general nature, it is weaker than the estimate of the form $F(T) \ll T\log^c T$, but is much stronger than Cramér's result $F(T) = O(T^{5/4+\varepsilon})$.

Example 2: The automorphic L-function attached to holomorphic cusp forms over $SL_2(\mathbb{Z})$. Let f be a primitive holomorphic cusp form of weight $k\ge 1$ for the full modular group $SL_2(\mathbb{Z})$. (For the sake of simplicity we only consider the form for $SL_2(\mathbb{Z})$.) Let

$$f(z) = \sum_{n=1}^{\infty}\lambda_f(n)n^{(k-1)/2}e(nz), \tag{10.1}$$

where $e(z) = e^{2\pi iz}$, be its normalized Fourier expansion at the cusp ∞. Then the automorphic L-function

$$\mathcal{L}(f,s) = \sum_{n=1}^{\infty}\lambda_f(n)n^{-s}$$

is an L-function of degree 2 satisfying the functional equation

$$(2\pi)^{-s}\Delta(s)\mathcal{L}(f,s) = (-1)^{k/2}(2\pi)^{-(1-s)}\Delta(1-s)\mathcal{L}(f,1-s)$$

with the gamma factor $\Delta(s) = \Gamma\left(s+\frac{k-1}{2}\right)$. Deligne [7] proved that $|\lambda_f(n)|\le d(n)$, so we have $\mathcal{L}(f,s)\in\mathcal{S}^0_{real}$. It is well-known that

$$A(y) = \sum_{n\le y}\lambda_f(n) \ll y^{1/3}$$

and (see Rankin [41] or Selberg [45])

$$\sum_{n \le y} |\lambda_f(n)|^2 = c_f y + O(y^{3/5})$$

for some positive constant c_f. Walfisz [53] proved that

$$\int_1^T A^2(y) dy = C_A T^{3/2} + O(T \log^2 T)$$

holds for some positive constant $C_A > 0$. From Corollary 1 we have a weaker result

$$\int_1^T A^2(y) dy = C_A T^{3/2} + O(T^{1+\varepsilon}).$$

Example 3: The automorphic L-function attached to Maass forms for $SL_2(\mathbb{Z})$.
Let φ be a primitive Maass form for $SL_2(\mathbb{Z})$, which is an eigenfunction of the Laplace operator with an eigenvalue $\lambda = 1/4 + r^2$, where $r \in \mathbb{R}$. Write its Fourier expansion at infinity in the form

$$\varphi(z) = \sqrt{v} \sum_{n \in \mathbb{Z} \setminus \{0\}} \rho(n) K_{ir}(2\pi |n| v) e(nu) \quad (\ z = u + iv, u \in \mathbb{R}, v > 0),$$

where K_{ir} is the modified Bessel function of the third kind. The corresponding automorphic L-function is

$$\mathcal{L}(\varphi, s) = \sum_{n=1}^{\infty} \rho(n) n^{-s},$$

which is a function of degree 2, entire on $\mathbb{C}$, with the functional equation

$$\pi^{-s} \Delta(s) \mathcal{L}(\varphi, s) = (-1)^{\delta} \pi^{-(1-s)} \Delta(1-s) \mathcal{L}(\varphi, 1-s),$$

where $\Delta(s) = \Gamma\left((s+\delta+ir)/2\right) \Gamma\left((s+\delta-ir)/2\right)$, and δ is the parity of φ defined by $\delta = 0$ if φ is even and $\delta = 1$ if φ is odd. Kim and Sarnak [30] proved that

$$|\rho(n)| \le d(n) n^{7/64}. \tag{10.2}$$

The Rankin-Selberg method implies that

$$\sum_{n \le y} |\rho(n)|^2 \ll y. \tag{10.3}$$

So we have $\mathcal{L}(\varphi, s) \in \mathcal{S}_{real}^{\theta}$ with $\theta = 7/64$.

Hafner and Ivić [14] proved that

$$A_r(y) = \sum_{n \le y} \rho(n) \ll y^{2/5}.$$

Meurman [37] proved that

$$A_r(y) \ll y^{\frac{1+\mu}{3}+\varepsilon},$$

where $\mu \geq 0$ is any real number such that $\rho(n) \ll n^{\mu}$.

From Corollary 1 we get

$$\int_1^T A_r^2(y)dy = C_{A_r}T^{3/2} + O(T^{1+\varepsilon}),$$

where $C_{A_r} > 0$ is a positive constant.

10.2 Some functions of degree 3

Example 1: Let $\mathcal{D}_r$ denote the set of all Dirichlet L-functions corresponding to real primitive Dirichlet characters, $\mathcal{D}_c$ denote the set of all Dirichlet L-functions corresponding to complex primitive Dirichlet characters, f denote a primitive holomorphic cusp form and φ denote a primitive Maass form for $SL_2(\mathbb{Z})$ for some fixed eigenvalue $\lambda = 1/4 + r^2 (r \in \mathbb{R})$, respectively. Define

$$\begin{aligned}
&\mathcal{D}_{1f} = \{\mathcal{L}(f,s)L(s,\chi):\ L(s,\chi) \in \mathcal{D}_r\},\\
&\mathcal{D}_{1\varphi} = \{\mathcal{L}(\varphi,s)L(s,\chi):\ L(s,\chi) \in \mathcal{D}_r\},\\
&\mathcal{D}_r^3 = \{L(s,\chi_1)L(s,\chi_2)L(s,\chi_3):\ L(s,\chi_j) \in \mathcal{D}_r, j = 1,2,3\},\\
&\mathcal{D}_r\mathcal{D}_c\overline{\mathcal{D}_c} = \{L(s,\chi_1)L(s,\chi_2)L(s,\overline{\chi_2}):\ L(s,\chi_1) \in \mathcal{D}_r, L(s,\chi_2) \in \mathcal{D}_c\}.
\end{aligned}$$

Let $\mathcal{L}(s)$ denote any function in the set $\mathcal{D}_{1f} \cup \mathcal{D}_{1\varphi} \cup \mathcal{D}^3 \cup \mathcal{D}_r\mathcal{D}_c\overline{\mathcal{D}_c}$. It is easy to see that $\mathcal{L}(s)$ is a function of degree 3, which satisfies the conditions of Corollary 2. We consider only the case $\mathcal{L}(s) = L(s,\chi)\mathcal{L}(\varphi,s)$, where χ is a real primitive Dirichlet character modulo some positive integer. Write

$$\mathcal{L}(s) = \sum_{n\geq 1} a(n)n^{-s},\ \operatorname{Re} s > 1.$$

Then $a(n) = \sum_{d|n}\rho(d)\chi(n/d)$. From the estimate (10.2) we have $a(n) \ll d^2(n)n^{7/64}$. By Cauchy's inequality we have

$$a^2(n) \leq d(n)\sum_{d|n}\rho^2(d),$$

which combining (10.3) gives

$$\sum_{n\leq y} a^2(n) \ll y^{\varepsilon/2}\sum_{dn\leq y}\rho^2(d) \ll y^{\varepsilon/2}\sum_{n\leq y}\sum_{d\leq y/n}\rho^2(d) \ll y^{1+\varepsilon},$$

where we used the bound $d(n) \ll n^{\varepsilon/2}$. Whence (1.3) holds.

As an analogue of the sixth moment of the automorphic L-function attached to a cusp form [25], we have also

$$\int_1^T |\mathcal{L}(\varphi, 1/2+it)|^6 dt \ll T^{2+\varepsilon}.$$

This estimate can be proved by Jutila's method analogously. So from Lemma 12 we have

$$\int_1^T |\mathcal{L}(7/12+it)|^2 dt \ll T^{1+\varepsilon},$$

namely, (1.4) holds for $\sigma^* = 7/12$, which obviously satisfies (1.5). So from Corollary 2 we have

$$\int_1^T E^2(y)dy = C_E T^{5/3} + O(T^{14/9+\varepsilon}),$$

where C_E is a positive constant.

Example 2: Let $K/\mathbb{Q}$ be a number field of degree κ with discriminant d_K. The Dedekind zeta-function defined by

$$\zeta_K(s) = \sum_{\mathfrak{a}\in\mathfrak{L}\setminus\{0\}} \frac{1}{(N\mathfrak{a})^s}$$

is an L-function of degree κ with the functional equation

$$(\pi^{-\kappa}|d_K|)^{s/2}\Delta(s)\zeta_K(s) = (\pi^{-\kappa}|d_K|)^{(1-s)/2}\Delta(1-s)\zeta_K(1-s)$$

with the gamma factor $\Delta(s) = \Gamma^{r_1+r_2}\left(\frac{s}{2}\right)\Gamma^{r_2}\left(\frac{s+1}{2}\right)$, where r_1 is the number of real embeddings of K and r_2 the number of pairs of complex embeddings so that $\kappa = r_1 + 2r_2$. The Dedekind zeta-function $\zeta_K(s)$ has a simple at $s = 1$.

From now we suppose that $\kappa = 3$. Let $E_K(y) := \sum_{n\le y} a(n) - \lambda y$, where $a(n)$ denotes the number of ideals in $\mathfrak{L}$ of norm n and λ is the residue of $\zeta_K(s)$ at $s = 1$. Weber and Landau showed that

$$E_K(y) = O\left(y^{1/2}\right).$$

In 1964, Chandrasekharan and Narasimhan [3] studied the mean square of $E_K(y)$ and proved

$$\int_1^T E_K^2(y)dy = O\left(T^{5/3}\log^3 T\right).$$

In 1999, Y.-K. Lau [33] improved the above result to

$$\int_1^T E_K^2(y)dy = O\left(T^{5/3}\log^2 T\right).$$

Fomenko [10] followed Tong's approach and proved that the asymptotic formula

$$\int_1^T E_K^2(y)dy = C_K T^{5/3} + O\left(T^{8/5+\varepsilon}\right)$$

holds for some positive constants $C_K > 0$ with the help of the estimate

$$\int_1^T |\zeta_K(\sigma+it)|^2 \ll T^{1+\varepsilon} \tag{10.4}$$

with $\sigma = 5/8$.

When $\kappa = 3$, we can show that the Dedekind zeta-function $\zeta_K(s)$ can be written as $\zeta_K(s) = \zeta(s)\mathcal{L}_2(s)$ such that where $\mathcal{L}_2(s) \in \mathcal{S}^0_{real}$ is a function of degree 2 satisfying the condition (1.9). When $K/\mathbb{Q}$ is a normal extension, from Lemma 1 of W. Müller [39] we know that $\zeta_K(s) = \zeta(s)L(s,\chi_1)L(s,\overline{\chi_1})$ for some primitive Dirichlet character χ_1. So we have $\zeta_K(s) = \zeta(s)\mathcal{L}_2(s)$, where $\mathcal{L}_2(s) = L(s,\chi_1)L(s,\overline{\chi_1})$ satisfies (1.9) via Meurman [38].

When $K/\mathbb{Q}$ is a not normal extension, from the formula (1) of Fomenko [10] we have $\zeta_K(s) = \zeta(s)L(s,F)$, where F is a holomorphic primitive cusp form of weight 1 with respect to $\Gamma_0(|d_K|)$ and $L(s,F)$ is its corresponding automorphic L-function of degree 2. We have the estimate

$$\int_1^T |L(1/2+it,F)|^6 dt \ll T^{2+\varepsilon}.$$

The proof is also due to Jutila [25], who proved it in the case of a holomorphic cusp form F of even weight with respect to $SL_2(\mathbb{Z})$. His proof can be applied to $L(s,F)$, too.

From Lemma 12 we see that (10.4) holds for $\sigma = 7/12$. So from Corollary 2 we get

$$\int_1^T E_K^2(y)dy = C_K T^{5/3} + O\left(T^{14/9+\varepsilon}\right),$$

which is a slight improvement to Fomenko's result.

10.3 Some examples of degree 4

Example 1. Let $\mathcal{D}_r$ denote the set of all Dirichlet L-functions corresponding to real primitive Dirichlet characters, $\mathcal{D}_c$ denote the set of all Dirichlet L-functions corresponding to complex primitive Dirichlet characters, f denote a primitive holomorphic cusp form and φ denote a primitive Maass form for $SL_2(\mathbb{Z})$ for some fixed an eigenvalue $\lambda = 1/4 + r^2 (r \in \mathbb{R})$, respectively. Define

$$\begin{aligned}
&\mathcal{D}_{2f} = \{\mathcal{L}(f,s)L(s,\chi_1)L(s,\chi_2):\ L(s,\chi_j) \in \mathcal{D}_r, \quad j=1,2\},\\
&\mathcal{D}_{2\varphi} = \{\mathcal{L}(\varphi,s)L(s,\chi_1)L(s,\chi_2):\ L(s,\chi_j) \in \mathcal{D}_r, \quad j=1,2\},\\
&\mathcal{D}_r^4 = \{\prod_{j=1}^4 L(s,\chi_j):\ L(s,\chi_j) \in \mathcal{D}_r, 1 \le j \le 4\},\\
&\mathcal{D}_r^2\mathcal{D}_c\overline{\mathcal{D}_c} = \{L(s,\chi_1)L(s,\chi_2)L(s,\chi_3)L(s,\overline{\chi_3}):\ L(s,\chi_1), L(s,\chi_2) \in \mathcal{D}_r, L(s,\chi_3) \in \mathcal{D}_c\},\\
&(\mathcal{D}_c\overline{\mathcal{D}_c})^2 = \{\prod_{j=1}^2 L(s,\chi_j)L(s,\overline{\chi_j}):\ L(s,\chi_j) \in \mathcal{D}_c, j=1,2\}.
\end{aligned}$$

Suppose $\mathcal{L}(s)$ is any function in the set

$$\mathcal{D}_{2f} \cup \mathcal{D}_{2\varphi} \cup \mathcal{D}^4 \cup \mathcal{D}_r^2\mathcal{D}_c\overline{\mathcal{D}_c} \cup (\mathcal{D}_c\overline{\mathcal{D}_c})^2 \cup \{\mathcal{L}^2(f,s), \mathcal{L}^2(\varphi,s), \mathcal{L}(f,s)\mathcal{L}(\varphi,s)\}.$$

Obviously $\mathcal{L}(s)$ is a function of degree 4. From Lemma 12 it is easy to see that

$$\int_1^T |\mathcal{L}(5/8+it)|^2 dt \ll T^{1+\varepsilon}. \tag{10.5}$$

For the mean square of $E(y)$ corresponding to $\mathcal{L}(s)$ we have

$$T^{7/4} \ll \int_1^T E^2(y)dy \ll T^{7/4+\varepsilon}. \tag{10.6}$$

The left inequality in (10.6) follows from Theorem 5 directly. The proof of the right inequality in (10.6) is the same as the case $k=4$ in Theorem 13.4 of Ivić [19] with the help of (10.5). So we omit the details. We note that the upper bound in (10.6) is already proved in Ivić [22] when $\mathcal{L}(s)=\mathcal{L}^2(f,s)$.

If we could find a number $1/2 \le \sigma_0 < 5/8$ such that

$$\int_1^T |\mathcal{L}(\sigma_0+it)|^2 dt \ll T^{1+\varepsilon},$$

then from Theorem 1 the asymptotic formula

$$\int_1^T E^2(y)dy = C_4 T^{7/4} + O(T^{7/4-\delta_{\mathcal{L}}}) \tag{10.7}$$

would hold, where $C_4>0$ and $\delta_{\mathcal{L}}>0$ are positive constants. The formula (10.7) implies that the estimate (10.6) is best possible.

Example 2: The error term in the Rankin-Selberg problem.

Let f be a primitive holomorphic cusp form of weight k for the full modular group $SL_2(\mathbb{Z})$ and its normalized Fourier expansion at the cusp ∞ can be written as (10.1). The corresponding Rankin-Selberg zeta function defined by

$$\mathcal{L}(f\otimes f,s)=\sum_{n=1}^{\infty} c(n)n^{-s},\ c(n)=\sum_{m^2|n} |\lambda_f(n/m^2)|^2,\ (\operatorname{Re} s>1)$$

is a function of degree 4 satisfying the functional equation

$$(2\pi)^{-2s}\Delta(s)\mathcal{L}(f\otimes f,s)=(2\pi)^{-2(1-s)}\Delta(1-s)\mathcal{L}(f\otimes f,1-s)$$

with the gamma factor $\Delta(s)=\Gamma(s)\Gamma(s+k-1)$.

Rankin [41] and Selberg [45] proved independently that

$$\sum_{n\le y} c(n) = C_f y + E(y),\ E(y)=O(y^{3/5}),$$

where $C_f>0$ is a positive constant. From Deligne's estimate $|\lambda_f(n)|\le d(n)$ we have

$$c(n)\le d^2(n)d(1,2;n)\le d^3(n)\ll n^{\varepsilon}, \tag{10.8}$$

where $d(1,2;n) = \sum_{n=m^2d} 1 \leq d(n)$. We write $c(n) = \sum_{n=m^2d} |\lambda_f(d)|^2$. By Cauchy's inequality we have

$$c^2(n) \leq d(1,2;n) \sum_{n=m^2d} |\lambda_f(d)|^4 \ll n^\varepsilon \sum_{n=m^2d} |\lambda_f(d)|^4,$$

where we used $d(1,2;n) \ll n^\varepsilon$. So we have

$$\begin{aligned} \sum_{n\leq y} c^2(n) &\ll y^\varepsilon \sum_{m^2d\leq y} |\lambda_f(d)|^4 \\ &\ll y^\varepsilon \sum_{m\leq y^{1/2}} \sum_{d\leq y/m^2} |\lambda_f(d)|^4 \\ &\ll y^\varepsilon \sum_{m\leq y^{1/2}} y/m^2 \ll y^{1+\varepsilon}, \end{aligned} \tag{10.9}$$

where we used the estimate in Lü [35]

$$\sum_{n\leq y} |\lambda_f(n)|^4 \ll y.$$

The estimates (10.8) and (10.9) show that (1.3) holds.

Ivić [21] proved that

$$\int_1^T E^2(y)dy \ll T^{1+2\beta+\varepsilon},$$

where $\beta = 2/(5-2\mu(1/2))$ with $\mu(1/2)$ satisfying $\zeta(1/2+it) \ll (|t|+1)^{\mu(1/2)+\varepsilon}$. Huxley's best bound[18] $\mu(1/2) \leq 32/205$ implies $\beta \leq 410/961 = 0.4266\cdots$. The Lindelöf Hypothesis implies $\beta \leq 0.4$. In [20] Ivić even conjectured

$$\int_1^T E^2(y)dy \ll T^{\frac{7}{4}+\varepsilon}. \tag{10.10}$$

The estimate (10.10), if it is true, is best possible. Since from Theorem 5 we have

$$\int_1^T E^2(y)dy \gg T^{\frac{7}{4}}.$$

If we could find a $\sigma_0 < \frac{5}{8}$ such that

$$\int_1^T |\mathcal{L}(f\otimes f, \sigma_0+it)|^2 dt \ll T^{1+\varepsilon},$$

then from Theorem 1 we would have the asymptotic formula

$$\int_1^T E^2(y)dy = C_f T^{\frac{7}{4}} + O\left(T^{\frac{7}{4}-\delta_f}\right)$$

for some positive constants $C_f > 0, \delta_f > 0$.

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